STOCHASTIC CONSUMPTION-SAVINGS MODEL: CANONICAL APPLICATIONS

SEPTEMBER 14, 2011

APPLICATIONS

- Use (solution to) stochastic two-period model to illustrate some basic results and ideas in
  - Consumption research
  - Asset pricing research
- Certainty-equivalent consumption
  - Assuming
    - Quadratic period-utility $u(c) = \gamma c - \frac{\alpha c^2}{2}$
    - Risk-free asset returns
    - Risky period-2 income (with arbitrary distribution)
- Risk aversion
- Precautionary savings
- Introduction to asset pricing

CERTAINTY EQUIVALENCE

- **Assume quadratic utility**

$$v(c_1,c_2) = u(c_1) + u(c_2) = \gamma c_1 - \frac{\alpha c_1^2}{2} + \gamma c_2 - \frac{\alpha c_2^2}{2}$$

- **Assume interest rate is not state contingent**

$$r_1^H = \bar{r}_1 = r_1^L = r_1 \qquad \text{risk-free interest rate}$$

- **Insert in definition of solution to intertemporal problem**

$$c_2^H + \underset{=0}{\cancel{a_2^H}} = y_2^H + (1+r_1)a_1 \qquad c_2^M + \underset{=0}{\cancel{a_2^M}} = \bar{y}_2 + (1+r_1)a_1 \qquad c_2^L + \underset{=0}{\cancel{a_2^L}} = y_2^L + (1+r_1)a_1 \qquad c_1 + a_1 = y_1 + (1+r_0)a_0$$

Euler eqn often the key $u'(c_1) = E_1\left[u'(c_2)(1+r_1)\right] \longrightarrow \gamma - \alpha c_1 = E_1\left[(\gamma - \alpha c_2)(1+r_1)\right]$

$$\longrightarrow \gamma - \alpha c_1 = q(\gamma - \alpha c_2^H)(1+r_1) + p(\gamma - \alpha c_2^M)(1+r_1) + (1-p-q)(\gamma - \alpha c_2^L)(1+r_1)$$

$$\longrightarrow \gamma - \alpha c_1 = (1+r_1)\left[q(\gamma - \alpha c_2^H) + p(\gamma - \alpha c_2^M) + (1-p-q)(\gamma - \alpha c_2^L)\right]$$

$$= (1+r_1)\left[\gamma - \alpha\underbrace{\left(qc_2^H + pc_2^M + (1-p-q)c_2^L\right)}_{=\,?}\right]$$

CERTAINTY EQUIVALENCE

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$$\longrightarrow \gamma - \alpha c_1 = (1+r_1)\left[q(\gamma - \alpha c_2^H) + p(\gamma - \alpha c_2^M) + (1-p-q)(\gamma - \alpha c_2^L)\right]$$

$$= (1+r_1)\left[\gamma - \alpha\underbrace{\left(qc_2^H + pc_2^M + (1-p-q)c_2^L\right)}\right]$$

$$\longrightarrow \gamma - \alpha c_1 = (1+r_1)\left[\gamma - \alpha E_1 c_2\right] \longrightarrow c_1 = -\frac{\gamma}{\alpha}r_1 + (1+r_1)E_1 c_2$$

CERTAINTY EQUIVALENCE

- **If not concerned with state-contingent solutions for c_2…**
- **…solution to consumer problem is an asset position and expected consumption profile $(c_1, E_1c_2; a_1)$ that satisfies**
  - **Period-2 budget constraint in expectation**

$$E_1c_2 = (1+r_1)a_1 + E_1y_2$$

  - **Euler equation**

$$c_1 = -\frac{\gamma}{\alpha}r_1 + (1+r_1)E_1c_2$$

  - **Period-1 budget constraint** $c_1 + a_1 = y_1 + (1+r_0)a_0$

taking as given $(r_1; y_1, a_0, r_0)$ and the stochastic distribution $G(.)$ of y_2

- **Optimal period-1 consumption**

$$c_1 = \underbrace{-\frac{\gamma}{\alpha}\left(\frac{r_1}{1+(1+r_1)^2}\right)}_{\equiv A} + \underbrace{\left(\frac{(1+r_1)^2}{1+(1+r_1)^2}\right)}_{\equiv B}\left(y_1+(1+r_0)a_0\right) + \underbrace{\left(\frac{1+r_1}{1+(1+r_1)^2}\right)}_{\equiv C}E_1y_2$$

CERTAINTY EQUIVALENCE

- **Optimal period-1 (current) consumption**

$$c_1 = A + B\cdot\left(y_1+(1+r_0)a_0\right) + C\cdot E_1y_2$$

  - **Depends only on the mean of risky future income, E_1y_2**
  - **Independent of second- and higher-moments of risky future income**

- **Distribution function $G(.)$ of period-2 income**

$$y_2 = \begin{cases} y_2^H & \text{probability q} \\ \bar{y}_2 & \text{probability p} \\ y_2^L & \text{probability 1-p-q} \end{cases} \quad \left.\begin{array}{l} E_1y_2 = \bar{y}_2 \\ \mathrm{Var}\, y_2 = q\left(y_2^H-\bar{y}_2\right)^2 + (1-p-q)\left(y_2^L-\bar{y}_2\right)^2 \end{array}\right.$$

- **Certainty Equivalence**
  - **Mean-preserving spreads of $G(.)$ do not affect optimal choice of c_1**
  - **E.g., (p = 1, q = 0)**
    - **Period-2 income has no risk**
    - **But c_1 is identical**
    - **s_1 (period-1 savings) is identical**

Certainty Equivalence

- A benchmark result in intertemporal consumption theory
- Result depends on
  - Quadratic utility
  - Riskless (aka non-state-contingent) asset returns
  - Only source of risk is income risk
- Only version of the intertemporal consumption model with analytical solution
- Strong implication: risk about future (income) does not affect current consumption and savings decisions
  - Intuitively plausible?
  - Empirically relevant?
  - Probably not…but why not?
- Model does feature both
  - Income risk (Var y_2 > 0)
  - Risk averse utility with respect to consumption – need to define formally

Applications

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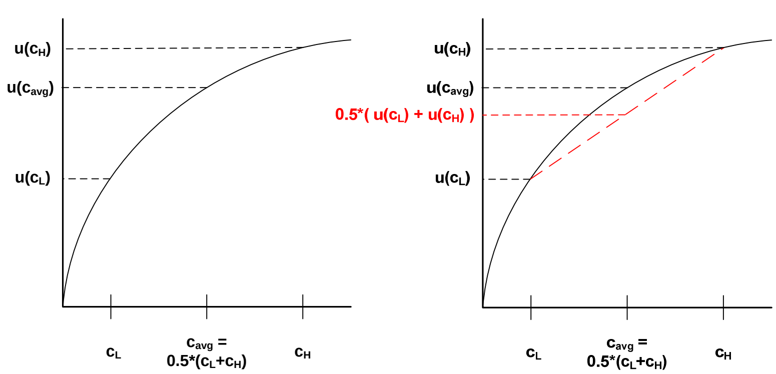
RISK AVERSION

- Illustrate with simple static example
- Utility function $u(c)$, with $u'(.) > 0$ and $u''(.) < 0$
- Two possible consumption outcomes
  - c^H with probability η
  - c^L with probability $1-\eta$
- Expected consumption is $\overline{c} = \eta c^H + (1-\eta)c^L$

RISK AVERSION



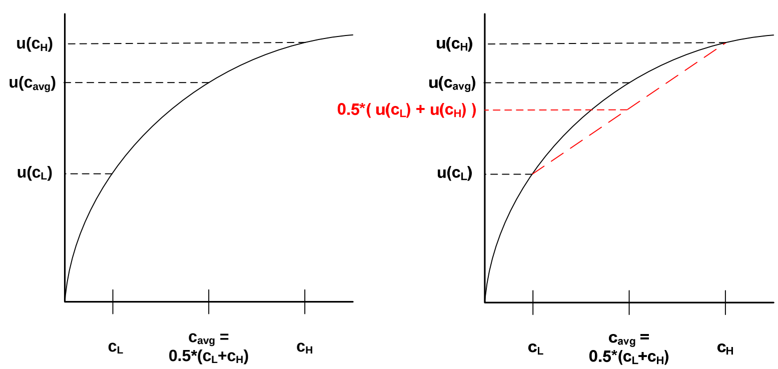
RISK AVERSION

- Illustrate with simple static example
- Utility function $u(c)$, with $u'(.) > 0$ and $u''(.) < 0$
- Two possible consumption outcomes
  - c^H with probability η
  - c^L with probability $1-\eta$
- Expected consumption is $\bar{c} = \eta c^H + (1-\eta)c^L$
- Definition: an individual is risk averse (with respect to consumption risk) if

$$u(\bar{c}) > \eta u(c^H) + (1-\eta)u(c^L)$$ JENSEN'S INEQUALITY

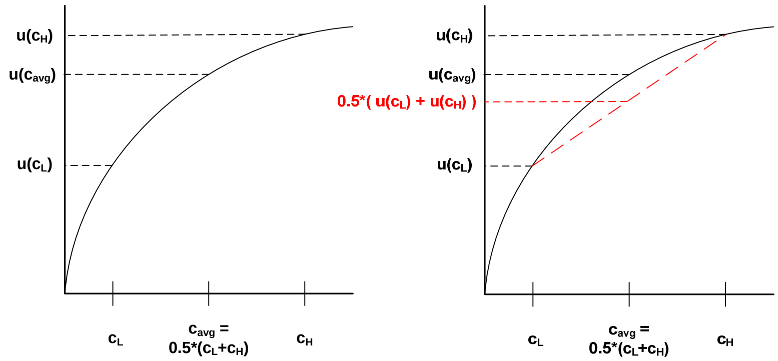
- Risk aversion
  - A preference for certain (deterministic) outcomes to risky (stochastic) outcomes
  - Embodied in strictly concave utility
- How to measure risk aversion?
  - Need to capture something about concavity of utility

RISK AVERSION



RISK AVERSION

- **How to measure?**



- **A candidate measure: $-u''(c)$**
  - **But not invariant to positive linear transformations of $u(.)$…**
  - **…even though implied choices are invariant to any monotonically increasing transformation of $u(.)$**

Macro/Finance Fundamentals

RISK AVERSION

- **Arrow-Pratt coefficient of absolute risk aversion (ARA)**

$$ARA(c) \equiv -\frac{u''(c)}{u'(c)}$$

Controls for linear transformations of $u(.)$

- **$ARA(c)$ gets at idea of risk aversion in level gains or losses of c from $E(c)$**
  - **Increasing ARA: $ARA'(c) > 0$**
  - **Decreasing ARA: $ARA'(c) < 0$**
    - **Most empirically-relevant case**
    - **"Richer people can afford to take a chance"**

Risk Aversion

- Arrow-Pratt coefficient of absolute risk aversion (ARA)

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Controls for linear transformations of $u(.)$

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    - Most empirically-relevant case
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- Perhaps also useful to have measure of risk aversion in percentage gains or losses of c from $E(c)$

- Relative risk aversion (RRA)

$$RRA(c) \equiv -\frac{cu''(c)}{u'(c)} \left(= c \cdot ARA(c)\right)$$

Adjusts for level of consumption / wealth

  - "Controlling for income / consumption, richer people cannot afford to take a chance anymore than anyone else"

Risk Aversion

- CRRA

$$v(c_1, c_2) = \underbrace{\frac{c_1^{1-\sigma} - 1}{1-\sigma}}_{\equiv u(c_1)} + \underbrace{\frac{c_2^{1-\sigma} - 1}{1-\sigma}}_{\equiv u(c_2)} \qquad \sigma > 0$$

Continuing to assume utility is additively-separable over time

- Attitude of consumers toward smoothing consumption between time periods
  - IES = $1/\sigma$

- Attitude of consumers toward risky outcomes within a given time period

$$RRA(c) = -\frac{cu''(c)}{u'(c)} = \ldots \qquad ARA(c) = -\frac{u''(c)}{u'(c)} = \ldots$$

RISK AVERSION

- CRRA

$$v(c_1,c_2) = \underbrace{\frac{c_1^{1-\sigma}-1}{1-\sigma}}_{\equiv u(c_1)} + \underbrace{\frac{c_2^{1-\sigma}-1}{1-\sigma}}_{\equiv u(c_2)} \qquad \sigma > 0$$

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- Attitude of consumers toward smoothing consumption between time periods
  - IES = 1/σ
- Attitude of consumers toward risky outcomes within a given time period

$$RRA(c) = -\frac{cu''(c)}{u'(c)} = ... \qquad ARA(c) = -\frac{u''(c)}{u'(c)} = ...$$

- CRRA utility: σ governs both intertemporal attitudes and intratemporal (relative) risk attitudes!
  - Inverses of each other!!
- Must/should IES and RRA be so directly related in reality?
  - Not at all…Epstein-Zin (EZ) utility function disentangles the two concepts

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PRECAUTIONARY SAVINGS

- **Certainty-equivalent consumption**
  - **Current consumption depends only on the mean of future risky income**
  - **Most important assumption: quadratic utility**
  - **Other necessary assumptions**
    - **Non-state-contingent asset returns**
    - **Future income the only source of risk**

PRECAUTIONARY SAVINGS

- **Certainty-equivalent consumption**
  - **Current consumption depends only on the mean of future risky income**
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- **Risk aversion (within period) with** $v(c_1,c_2)=u(c_1)+u(c_2)=\gamma c_1-\frac{\alpha c_1^2}{2}+\gamma c_2-\frac{\alpha c_2^2}{2}$ **?**
  - **Obviously $\neq$ 0! (whether RRA or ARA)**
  - **So why certainty equivalence?**
    i.e., why does future income risk "not matter" for current choices?

Euler eqn often the key $u'(c_1)=E_1\left[u'(c_2)(1+r_1)\right] \longrightarrow \gamma-\alpha c_1=E_1\left[(\gamma-\alpha c_2)(1+r_1)\right]$

$$\longrightarrow \gamma-\alpha c_1=q(\gamma-\alpha c_2^H)(1+r_1)+p(\gamma-\alpha c_2^M)(1+r_1)+(1-p-q)(\gamma-\alpha c_2^L)(1+r_1)$$

$$\longrightarrow \gamma-\alpha c_1=(1+r_1)\left[q(\gamma-\alpha c_2^H)+p(\gamma-\alpha c_2^M)+(1-p-q)(\gamma-\alpha c_2^L)\right]$$
$$=(1+r_1)\left[\gamma-\alpha\underbrace{\left(qc_2^H+pc_2^M+(1-p-q)c_2^L\right)}\right]$$

$$\longrightarrow \gamma-\alpha c_1=(1+r_1)\left[\gamma-\alpha E_1c_2\right] \longrightarrow c_1=-\frac{\gamma}{\alpha}r_1+(1+r_1)E_1c_2$$

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$$=(1+r_1)\left[\gamma-\alpha\underbrace{\left(qc_2^H+pc_2^M+(1-p-q)c_2^L\right)}\right]$$

Because of linear *marginal* utility!!!

$$\longrightarrow \gamma-\alpha c_1=(1+r_1)\left[\gamma-\alpha E_1c_2\right] \longrightarrow c_1=-\frac{\gamma}{\alpha}r_1+(1+r_1)E_1c_2$$

PRECAUTIONARY SAVINGS

- **Certainty-equivalent consumption**
  - **Current consumption depends only on the mean of future risky income**
  - **Most important assumption: quadratic utility**
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  - **Obviously $\not\approx$ 0! (whether RRA or ARA)**
  - **So why certainty equivalence?**
- **Marginal utility function of order one (or lower) *implies* risk on future income doesn't matter for current consumption**

  $\downarrow$ **Contrapositve**
- **Risk on future income matters for current consumption *implies* marginal utility function must be strictly convex**
- ***u'''(c)* > 0 necessary for breaking certainty-equivalence result**
  - **(Given *u'*(.) > 0 and *u''*(.) < 0)**
  - ***u'''*(.) > 0 → *u''*(.) increasing in *c* → *u'*(.) decreasing less quickly as *c*↑**
  - **Not satisfied by quadratic utility**

PRECAUTIONARY SAVINGS

- **Assume utility with $u'''(c) > 0$**

$$v(c_1, c_2) = u(c_1) + u(c_2)$$

- **Assume interest rate is not state contingent**

$$r_1^H = \bar{r}_1 = r_1^L = r_1$$

risk-free interest rate

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- **Insert in definition of solution to intertemporal problem**

$$c_2^H + \underbrace{a_2^H}_{=0} = y_2^H + (1+r_1)a_1 \qquad c_2^M + \underbrace{a_2^M}_{=0} = \bar{y}_2 + (1+r_1)a_1 \qquad c_2^L + \underbrace{a_2^L}_{=0} = y_2^L + (1+r_1)a_1 \qquad c_1 + a_1 = y_1 + (1+r_0)a_0$$

Euler eqn often the key $u'(c_1) = E_1\left[u'(c_2)(1+r_1)\right] \longrightarrow u'(c_1) = (1+r_1)E_1\left[u'(c_2)\right]$

$$\longrightarrow u'(c_1) = (1+r_1)\left[\underbrace{qu'(c_2^H) + pu'(c_2^M) + (1-p-q)u'(c_2^L)}\right]$$

$\neq E_1c_2$, so none of the subsequent steps with quadratic $u(.)$ follow

- **$u'''(c) > 0$ → current consumption depends on distribution $G(.)$ of future risk**
  - **i.e., on first- and (in principle) all higher-order moments of $G(.)$**

PRECAUTIONARY SAVINGS

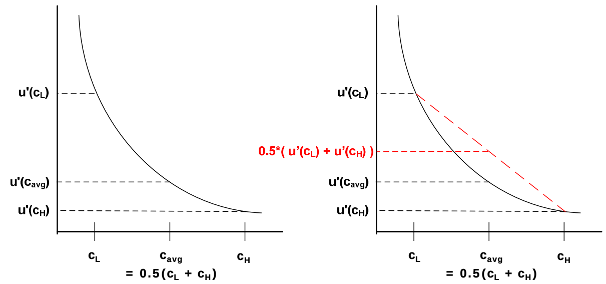
- $u'''(c) > 0$ → current consumption depends on distribution $G(.)$ of future risk
- Is optimal c_1 larger or smaller than certainty-equivalent c_1?
  - For a given $G(.)$

PRECAUTIONARY SAVINGS

- $u'''(c) > 0$ → current consumption depends on distribution $G(.)$ of future risk
- Optimal c_1 is smaller than certainty-equivalent c_1
  - Proof:
- Implication: optimal s_1 is larger than certainty-equivalent s_1
- Precautionary Savings
  - Risk about the future induces prudent (cautious) choices in the present
  - Desire to build up a buffer stock of assets to ensure c does not fall too low in future
  - Risk aversion a necessary, but not sufficient, feature of preferences
  - Strictly convex marginal utility the key feature of preferences
  - Classic papers: Kimball (1990 *Econometrica*), Sandmo (1970 *Review of Economic Studies*)
- How to measure precautionary savings motive?
  - Need to capture something about convexity of marginal utility
  - Kimball (1990) provides clever insight

PRECAUTIONARY SAVINGS

- How to measure?



- **A candidate measure: $u'''(c)$**
  - Analogy with measures of risk aversion

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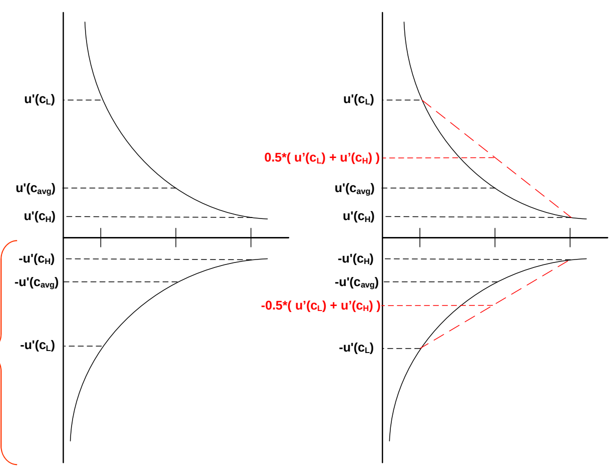

Kimball (1990):
Define $v(c) = -u'(c)$.
Then can apply standard theory of risk aversion to $v(c)$!

PRECAUTIONARY SAVINGS

- Coefficient of absolute prudence: $-\frac{u'''(c)}{u''(c)}$
- Coefficient of relative prudence: $-\frac{cu'''(c)}{u''(c)}$
- Measures of the sensitivity of optimal choice to risk
  - Governed by marginal utility function
- ARA and RRA measure the sensitivity of welfare to risk
  - Governed by the utility function

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- CRRA utility

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \rightarrow -\frac{u'''(c)}{u''(c)} = \frac{\sigma+1}{c} \quad \text{Absolute prudence}$$

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \rightarrow -\frac{cu'''(c)}{u''(c)} = \sigma+1 \quad \text{Relative prudence}$$

  - Displays constant relative prudence
  - Displays constant relative risk aversion

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ASSET MARKETS

- Risk about the future (period 2) requires adopting a view about the nature of asset markets
- Continue with example of risky period-2 income

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ASSET MARKETS

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- **But now three "distinct" assets available for purchase in period 1**
  - **Asset a^H: purchase price R^H in period 1, pays off one unit in period 2 if y_2^H, zero else**
  - **Asset a^M: purchase price $\bar{R}$ in period 1, pays off one unit in period 2 if $\bar{y}_2$, zero else**
  - **Asset a^L: purchase price R^L in period 1, pays off one unit in period 2 if y_2^L, zero else**
- **Arrow-Debreu securities, aka contingent claims**
- **NOT EQUIVALENT to state-contingent asset returns on a *single* asset**

STATE-CONTINGENT CHOICES

- **Consumer problem**

$$\max u(c_1) + qu(c_2^H) + pu(c_2^M) + (1-p-q)u(c_2^L) + \lambda_1\left[y_1 + a_0 - c_1 - R^H a_1^H - \bar{R}a_1^M - R^L a_1^L\right]$$
$$+q\lambda_2^H\left[y_2^H + a_1^H - c_2^H\right] + p\lambda_2^M\left[\bar{y}_2 + a_1^M - c_2^M\right] + (1-p-q)\lambda_2^L\left[y_2^L + a_1^L - c_2^L\right]$$

- **FOCs**

BASICS OF ASSET PRICING

- **Consumer problem**

$$\max u(c_1)+qu(c_2^H)+pu(c_2^M)+(1-p-q)u(c_2^L)+\lambda_1\left[y_1+a_0-c_1-R^H a_1^H-\bar{R}a_1^M-R^L a_1^L\right]$$
$$+q\lambda_2^H\left[y_2^H+a_1^H-c_2^H\right]+p\lambda_2^M\left[\bar{y}_2+a_1^M-c_2^M\right]+(1-p-q)\lambda_2^L\left[y_2^L+a_1^L-c_2^L\right]$$

- **FOCs**
- **Asset prices**

$$R^H=\frac{q\lambda_2^H}{\lambda_1}=\frac{qu'(c_2^H)}{u'(c_1)} \qquad \bar{R}=\frac{p\lambda_2^M}{\lambda_1}=\frac{pu'(c_2^M)}{u'(c_1)} \qquad R^L=\frac{(1-p-q)\lambda_2^L}{\lambda_1}=\frac{(1-p-q)u'(c_2^L)}{u'(c_1)}$$

  - $u'(c_2^j)/u'(c_1)$ **is willingness to intertemporally substitute consumption between period 1 and state *j* in period 2 – intertemporal MRS (IMRS)**

- **Contingent claims prices (aka Arrow-Debreu prices, aka state prices) reflect IMRS (if markets functioning well)**

- **In principle, allow for inferences about**
  - **Risk aversion**
  - **Prudence**
  - **Market participants' assessment of probabilities of event *j* occurring**

BASICS OF ASSET PRICING

- **Generalize the period-2 risk structure**
  - ***S*: number of possible realizations of y_2 (in richer models, risk in other primitives)**
  - **R^j: period-1 price of AD security that pays off one unit in state *j*, zero otherwise**
  - **p^j: probability of state *j* occurring in period 2, with** $\sum_{j=1}^{S} p^j = 1$

- **Lifetime expected utility** $u(c_1)+E_1 u(c_2)=u(c_1)+\sum_{j=1}^{S} p^j u(c_2^j)$
- **Period-1 budget constraint** $c_1+\sum_{j=1}^{S} R^j a_1^j = y_1+a_0$
- **State-*j* period-2 budget constraint** $c_2^j+\underbrace{a_2^j}_{=0}=y_2^j+a_1^j,\quad j\in\{1,2,3,\dots,S\}$
- **AD price for state *j* (compute FOCs)**

BASICS OF ASSET PRICING

- Generalize the period-2 risk structure
  - ***S***: number of possible realizations of y_2 (in richer models, risk in other primitives)
  - ***R^j***: period-1 price of AD security that pays off one unit in state ***j***, zero otherwise
  - ***p^j***: probability of state j occurring in period 2, with $\sum_{j=1}^{S} p^j = 1$
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- State-***j*** period-2 budget constraint $c_2^j + \underbrace{a_2^j}_{=0} = y_2^j + a_1^j, \quad j \in \{1, 2, 3, ..., S\}$
- AD price for state ***j*** (compute FOCs)

$$R^j = \frac{p^j \lambda_2^j}{\lambda_1} = \frac{p^j u'(c_2^j)}{u'(c_1)}$$

- Define $R^f \equiv \sum_{j=1}^{S} R^j = \sum_{j=1}^{S} p^j \frac{u'(c_2^j)}{u'(c_1)} = E_1\left[\frac{u'(c_2)}{u'(c_1)}\right]$
  - Is the price of a one-period riskless bond

Stochastic Consumption-Savings Model: Application

BASICS OF ASSET PRICING

- One-period riskless bond
  - Purchase price ***R^f*** in period 1
  - Pays off one unit ("face value") in all states of the world in period 2
  - (Can scale to any arbitrary face value: \$100 bonds, \$1000 bonds, etc.)
- Introduce in model
  - Period-1 budget constraint $c_1 + R^f b_1 + \sum_{j=1}^{S} R^j a_1^j = y_1 + a_0$
- State-***j*** period-2 budget constraint $c_2^j + \underbrace{b_2^j}_{=0} + \underbrace{a_2^j}_{=0} = y_2^j + b_1 + a_1^j, \quad j \in \{1, 2, 3, ..., S\}$

***b*₁**: bond holdings carried from period 1 to period 2

- FOC on b_1

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b_1: bond holdings carried from period 1 to period 2

- **FOC on b_1**

$$R^f = E_1\left[\frac{\lambda_2}{\lambda_1}\right] = E_1\left[\frac{u'(c_2)}{u'(c_1)}\right]$$

Price of riskless bond reflects expected IMRS…

$$= \sum_{j=1}^{S} p^j \frac{u'(c_2^j)}{u'(c_1)} = \sum_{j=1}^{S} R^j$$

…and by no-arbitrage equals sum of state prices.

- **Result: risk-free bond price can be decomposed into state prices**
  - **A complete set of AD securities spans the risk space…**
  - **…which makes b_1 a redundant asset; consumer can synthesize b_1 himself**
- **Any asset can be decomposed into state prices (Cochrane, Chapter 3.1)**

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- **How do these asset structures affect consumer's intertemporal life?**

CONSUMPTION, SAVINGS, AND ASSET PRICES

- **Consumption smoothing a primitive feature of preferences ($u'(.)>0$, $u''(.)<0$)**
- **Nature of asset markets affects ability to achieve consumption smoothing**
- **Two dimensions of consumption smoothing**
- **Intertemporal consumption smoothing: concavity of $u(.)$ implies preference for *low time-series-variance* of consumption**

$$R^f = E_1\left[\frac{u'(c_2)}{u'(c_1)}\right]$$

Expected IMRS = price of risk-free bond

$$\Leftrightarrow R^f u'(c_1) = E_1 u'(c_2)$$

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- **Intratemporal consumption smoothing: concavity of $u(.)$ implies preference for *low cross-state variance* of consumption *within any period that has risk***

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- **A high state price R^j reflects**
  - **High probability of state *j***
  - **High *u*'(.) in state *j* – i.e., low consumption in state *j***
  - **Or both**

- **View as intratemporal optimality condition across future state-contingent *c***

$$\frac{R^j / p^j}{R^k / p^k} = \frac{u'(c_2^j)}{u'(c_2^k)}, \qquad \forall j,k \in \{1,2,3,...,S\}$$

MRS across states *j*, *k* = (risk-adjusted) relative state price

CONSUMPTION, SAVINGS, AND ASSET PRICES

- **Define $m^j = R^j / p^j$ as discount factor for state *j***

- **Intratemporal optimality condition**

$$\frac{m^j}{m^k} = \frac{u'(c_2^j)}{u'(c_2^k)}, \qquad \forall j,k \in \{1,2,3,...,S\}$$

- **Intertemporal optimality between period 1 and state *j* in period 2**

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$$m^j = \frac{u'(c_2^j)}{u'(c_1)}, \qquad \forall j \in \{1,2,3,\ldots S\}$$

- **Expected IMRS between period 1 and period 2**

Terminology:
Stochastic discount factor (SDF)

$$E_1 m \equiv \sum_{j=1}^{S} p^j m^j$$

$$= R^f = E_1\left[\frac{u'(c_2)}{u'(c_1)}\right]$$