STOCHASTIC CONSUMPTION-SAVINGS MODEL: FURTHER APPLICATIONS

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APPLICATIONS

- Use (solution to) stochastic two-period model to illustrate some basic results and ideas in
  - Consumption research
  - Asset pricing research
- Subjective discount factor
- Lifetime "tilt" of consumption
- Competitive pricing of Arrow-Debreu assets
- Consumption-smoothing revisited
- Fiscal policy applications
  - Ricardian Equivalence
  - Consumption Taxation

SUBJECTIVE DISCOUNT FACTOR

- Impatience potentially an issue when taking a serious view of time
- Individuals (i.e., consumers) are impatient
  - All else equal, would rather have outcome *X* today than identical outcome *X* at some future date
  - An introspective statement about the world
  - An empirical statement about the world
- Subjective discount factor
  - A simple model of individuals' impatience about utility
  - β (a scalar in [0,1]) measures impatience
    - The smaller is β, the less does individual value future utility

Given $u'(.) > 0$, impatience about future c

Macro Fundamentals

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Given $u'(.) > 0$, impatience about future c

- Simple assumption about how "impatience" builds up over time
  - Multiplicatively: i.e., discount one period ahead by β, discount two periods ahead by β^2, discount three periods ahead by β^3, etc.
- Do individuals' impatience really build up over time in this geometric way?
  - Hyperbolic discounting likely a better model (later)

EXPECTED UTILITY

- Preferences $v(c_1, c_2)$ with all the "usual properties"
  - Separable across time
  - Strictly increasing in each of c_1 and c_2
  - Diminishing marginal utility in each of c_1 and c_2

- Expected lifetime ***discounted*** utility

$$v(c_1,c_2) = u(c_1) + \beta E_1 u(c_2) \left(= u(c_1) + \beta q u(c_2^H) + \beta p u(c_2^M) + \beta(1-p-q)u(c_2^L)\right)$$

  - Treating β as time-invariant constant
  - Modern business-cycle models sometimes treat it as stochastic
    - (Microfoundations?...)

- Beyond two-period planning horizon

$$u(c_1) + \beta E_1 u(c_2) + \beta^2 E_1 u(c_3) + \beta^3 E_1 u(c_4) + \beta^4 E_1 u(c_5) + \ldots$$

STATE-CONTINGENT CHOICES

- Sequential Lagrange formulation

$$\max u(c_1) + \beta E_1 u(c_2) + \lambda_1 \left[y_1 + (1+r_0)a_0 - c_1 - a_1 \right] + \beta E_1 \left\{ \lambda_2 \left[y_2 + (1+r_1)a_1 - c_2 \right] \right\}$$

- FOCs

(1) c_1: $u'(c_1) - \lambda_1 = 0$ — Marginal value of period-1 resources = marginal utility of c_1

(2) a_1: $-\lambda_1 + \beta E_1 \{\lambda_2 (1+r_1)\} = 0$

(3) c_2: $\beta E_1 u'(c_2) - \beta E_1 \lambda_2 = 0$

  - Note that (3) is not the period-2 statement of (1)

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- **Sequential Lagrange formulation**

$$\max u(c_1)+\beta E_1 u(c_2)+\lambda_1\left[y_1+(1+r_0)a_0-c_1-a_1\right]+\beta E_1\left\{\lambda_2\left[y_2+(1+r_1)a_1-c_2\right]\right\}$$

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(3) c_2: $\beta E_1 u'(c_2)-\beta E_1\lambda_2=0$

  - **Note that (3) is not the period-2 statement of (1)**

- **But with state-contingent asset returns, condition (1) does hold in period 2**
  - **Write out expectations in optimization explicitly**

$$\max u(c_1)+\lambda_1\left[y_1+(1+r_0)a_0-c_1-a_1\right]+\beta q u(c_2^{\,H})+\beta q\lambda_2^{\,H}\left[y_2^{\,H}+(1+r_1^{\,H})a_1-c_2^{\,H}\right]$$
$$+\beta p u(c_2^{M})+\beta p\lambda_2^{M}\left[\overline{y}_2+(1+\overline{r}_1)a_1-c_2^{M}\right]+\beta(1-p-q)u(c_2^{L})$$
$$+\beta(1-p-q)\lambda_2^{L}\left[y_2^{\,L}+(1+r_1^{\,L})a_1-c_2^{\,L}\right]$$

↓ FOCs

STATE-CONTINGENT CHOICES

- **FOCs**

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(2) a_1: $-\lambda_1+\beta E_1\{\lambda_2(1+r_1)\}=0$ — Euler equation

(3)
$$\begin{cases} c_2^H: & \beta u'(c_2^{\,H})-\beta\lambda_2^{\,H}=0 \\ c_2^M: & \beta u'(c_2^{M})-\beta\lambda_2^{M}=0 \\ c_2^L: & \beta u'(c_2^{L})-\beta\lambda_2^{L}=0 \end{cases}$$

Marginal value of period-2 resources = marginal utility of c_2
IMPORTANT: Holds state-by-state
(and thus also holds in expectation)

- **Analyze (2)**
  - **Express as an asset-pricing condition** $1=E_1\left\{\frac{\beta\lambda_2}{\lambda_1}(1+r_1)\right\}$

Price in period 1 of purchasing one unit of a_1

Expected discounted return in period 2
Note covariance between (λ_2/λ_1) and ($1+r_1$)

STATE-CONTINGENT CHOICES

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  - **Express as an asset-pricing condition** $1 = E_1\left\{\frac{\beta\lambda_2}{\lambda_1}(1+r_1)\right\}$
    - Price in period 1 of purchasing one unit of a_1
    - Expected discounted return in period 2
    - Note covariance between (λ_2/λ_1) and $(1+r_1)$
  - **Or write out expectations in (2) explicitly**

$$-\lambda_1 + \beta q\lambda_2^H(1+r_1^H) + \beta p\lambda_2^M(1+\bar{r}_1) + \beta(1-p-q)\lambda_2^L(1+r_1^L) = 0 \quad \longleftrightarrow \quad 1 = \frac{\beta q\lambda_2^H}{\lambda_1}(1+r_1^H) + \frac{\beta p\lambda_2^M}{\lambda_1}(1+\bar{r}_1) + \frac{\beta(1-p-q)\lambda_2^L}{\lambda_1}(1+r_1^L)$$

The prices of Arrow-Debreu securities.
Terminology: "state prices"

CONSUMPTION DYNAMICS

- **Solution to consumer problem is an asset position and state-contingent consumption profile** $\left(c_1, c_2^H, c_2^M, c_2^L; a_1\right)$ **that satisfies**
  - **State-by-state period-2 budget constraint**

5 equations, 5 unknowns
In principle, can solve

$$c_2^H + a_2^H = y_2^H + (1+r_1^H)a_1 \qquad c_2^M + a_2^M = \bar{y}_2 + (1+\bar{r}_1)a_1 \qquad c_2^L + a_2^L = y_2^L + (1+r_1^L)a_1$$

($a_2^H = 0$, $a_2^M = 0$, $a_2^L = 0$)

  - **Euler equation**

$$1 = \frac{\beta q u'(c_2^H)}{u'(c_1)}(1+r_1^H) + \frac{\beta p u'(c_2^M)}{u'(c_1)}(1+\bar{r}_1) + \frac{\beta(1-p-q)u'(c_2^L)}{u'(c_1)}(1+r_1^L) \left(= \beta E_1\left[\frac{u'(c_2)}{u'(c_1)}(1+r_1)\right]\right)$$

  - **Period-1 budget constraint** $c_1 + a_1 = y_1 + (1+r_0)a_0$

taking as given $\left(r_1^H, \bar{r}_1, r_1^L; y_1, a_0, r_0\right)$ **and the stochastic distribution** $G(.)$ **of** y_2

- **Could express solution in alternative ways**
  - **e.g., using lifetime budget constraints**

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LIFECYCLE CONSUMPTION PROFILE

- Suppose no income risk and no interest rate risk

$$y_2^H = \bar{y}_2 = y_2^L = y_2 \qquad r_1^H = \bar{r}_1 = r_1^L = r_1$$

  - i.e., back to deterministic case
- Insert in definition of solution to intertemporal problem
- Solution to consumer problem is an asset position and consumption profile (c_1, c_2, a_1) that satisfies
  - Period-1 budget constraint $c_1 + a_1 = y_1 + (1+r_0)a_0$
  - Period-2 budget constraint $c_2 + \underbrace{a_2}_{=0} = y_2 + (1+r_1)a_1$
  - Euler equation $u'(c_1) = \beta u'(c_2)(1+r_1)$ aka consumption-savings optimality condition

  taking as given $(r_1; y_1, y_2, a_0, r_0)$
- IMRS = price representation of Euler equation $\dfrac{u'(c_1)}{\beta u'(c_2)} = 1 + r_1$

LIFECYCLE CONSUMPTION PROFILE

- Tilt of lifetime consumption depends on $\beta(1+r_1)$

$$\frac{u'(c_1)}{\beta u'(c_2)} = 1 + r_1$$

- $\beta(1+r_1) = 1$
- $\beta(1+r_1) < 1$
- $\beta(1+r_1) > 1$

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ASSET MARKETS

- Risk about the future (period 2) requires adopting a view about the nature of asset markets
- Continue with example of risky period-2 income

$$y_2 = \begin{cases} y_2^H & \text{probability q} \\ \bar{y}_2 & \text{probability p} \\ y_2^L & \text{probability 1-p-q} \end{cases}$$

- But now three "distinct" assets available for purchase in period 1
  - Asset a^H: purchase price R^H in period 1, pays off one unit in period 2 if y_2^H, zero else
  - Asset a^M: purchase price $\bar{R}$ in period 1, pays off one unit in period 2 if $\bar{y}_2$, zero else
  - Asset a^L: purchase price R^L in period 1, pays off one unit in period 2 if y_2^L, zero else
- Arrow-Debreu securities, aka contingent claims
- Equivalent to state-contingent asset returns on a "single" asset

COMPETITIVE ASSET PRICING

- Consumer optimization yields

$$R^j = \frac{\beta p^j \lambda_2^j}{\lambda_1} = \frac{\beta p^j u'(c_2^j)}{u'(c_1)}, \qquad j \in \{1, 2, 3, \ldots, S\}$$

  - Demand-side asset pricing relations for each Arrow-Debreu asset
- Suppose perfectly-competitive sellers of Arrow-Debreu securities
  - A representative financial intermediary

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- (Expected) profit-maximization problem of intermediary (solved in period 1)
  - Taking prices R^j as given

$$\max_{\{a_1^j\}_{j=1}^S} \left(\sum_{j=1}^S R^j a_1^j - \sum_{j=1}^S p^j a_1^j \right)$$

Total revenue = sales of each AD asset

Total expected cost = expected payout in period 2 (no transactions etc. costs)

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$$R^j = p^j, \qquad j \in \{1,2,3,...,S\}$$

- Result: competitively-priced Arrow-Debreu securities have price = probability of event

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- Definition: $m^j = R^j / p^j$ is discount factor for state j

$$\Rightarrow m^j = 1 \quad \forall j$$ under competitive AD markets

- Insurance applications (Econ 603/604)
  - "Actuarially-fair insurance" – interpretable as complete AD markets

CONSUMPTION SMOOTHING REVISITED

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$$\Rightarrow m^j = 1 \quad \forall j$$ under competitive AD markets

- Intratemporal optimality condition

$$\frac{m^j}{m^k} = \frac{u'(c_2^j)}{u'(c_2^k)}, \qquad \forall j, k \in \{1, 2, 3, ..., S\}$$ under competitive AD markets

CONSUMPTION SMOOTHING REVISITED

- **Definition: $m^j = R^j / p^j$ is discount factor for state *j***

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- **Intratemporal consumption-smoothing**

$$u'(c_2^j) = u'(c_2^k) \quad \Rightarrow \quad c_2^j = c_2^k \qquad \forall j,k \in \{1,2,3,...,S\}$$ **under competitive AD markets**

Stochastic Consumption-Savings Model: Application

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  - **State-contingent period-2 consumption equated across all states**

- **Period-2 income risk fully insured away!**

  - **Period-2 consumption the same no matter the realization of risk**

$$\Rightarrow \quad c_2 = c_2^j = c_2^k = E_1 c_2, \quad \forall j,k \in \{1,2,3,...,S\}$$ **with certainty!**

  - **"Full consumption insurance"**
    - **For a given individual, across possible future states of the world**

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↓ In turn has consequences for intertemporal consumption smoothing

CONSUMPTION SMOOTHING REVISITED

- **Intertemporal optimality condition**

$$E_1\left[\frac{\beta u'(c_2)}{u'(c_1)}\right] = \sum_{j=1}^{S} \frac{p^j \beta u'(c_2^j)}{u'(c_1)} = \frac{\beta u'(c_2)}{u'(c_1)} \sum_{j=1}^{S} p^j = \frac{\beta u'(c_2)}{u'(c_1)} = R^f$$

↑ because perfect cross-state smoothing in period 2

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because perfect cross-state smoothing in period 2

- Now simply a deterministic consumption-savings optimality condition!

$$\frac{\beta u'(c_2)}{u'(c_1)}=R^f \Leftrightarrow \frac{u'(c_1)}{\beta u'(c_2)}=\frac{1}{R^f}\equiv 1+r_1$$

  - R^f: price of risk-free bond

Define as riskless return (on one-period bond)

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Define as riskless return (on one-period bond)

- If $\beta(1+r_1) = 1$

$$c_1=c_2=c_2^j=c_2^k, \qquad \forall j,k\in\{1,2,3,...,S\}$$

  - Consumption smoothing across all dates and all states!

CONSUMPTION AND ASSET PRICING THEORY

- Intertemporal consumption model the backbone of modern macro models
  - Business cycle models
  - Growth models
  - An application of basic consumer theory
- Basic issues / results / questions
  - Intertemporal substitution of consumption
  - Risk aversion
  - Precautionary savings
  - Consumption smoothing (over dates and over states) via financial assets
  - Subjective discount factor
- Asset pricing can be (but need not be…) studied as the flip side of intertemporal consumption theory

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- Empirical validity of baseline intertemporal consumption model?
- Empirical validity of baseline asset pricing theory?
- Next
  - Extension to infinite horizon
  - Introduction to dynamic programming