INFINITE-HORIZON
CONSUMPTION-SAVINGS MODEL

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Introduction

Quantitative macro models feature an infinite number of periods
   A more realistic (?) view of time

Infinite number of periods
   A metaphor for “many periods”
   A metaphor for altruistic linkages between generations
   Mathematically convenient because recursive representation

Limit of a \( T \)-period economy as \( T \to \infty \)

One-period subjective discount factor \( \beta \) (a scalar in \([0,1]\))
   With infinity of periods, can interpret as simple way of modeling finite planning horizon

“Impatience” aggregates geometrically over time
   Subjective discount for \( S \) periods ahead is \( \beta^S \)

Typically begin analysis from beginning of period zero
   Index time with arbitrary indexes \( t, t+1, t+2, \ldots \)
BASICS

- Recast/restate (some) consumption theory results
- Recast/restate (some) financial theory results
- New technical issues
- State-contingent decisions (still) the key

- Describe as an optimal decision rule
  \[ c_t(y_t) \]

- A mapping from realized income state to optimal choice in that state

- Infinite-horizon analog: \( c_t(y_t) \)
  - But a more complicated object because branching of event tree

UTILITY

- Preferences \( v(c_{0t}, c_1, c_2, \ldots) \) with all the “usual properties”
  - Separable across time
  - Strictly increasing in each \( c_j \) individually
  - Diminishing marginal utility in each of \( c_j \) individually

- Expected lifetime discounted utility

\[
E_0 v(c_0, c_1, c_2, \ldots) = u(c_0) + \beta E_0 u(c_1) + \beta^2 E_0 u(c_2) + \beta^3 E_0 u(c_3) + \ldots
= E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)
\]

- “Planning time” aligned with “calendar time” by starting at period zero

- Conditional on information set of period zero

- Later consider alternative formulations (within the class of expected utility and rational expectations)
  - Time-non-separable utility
  - Non-geometric subjective discounting
**UTILITY**

- Preferences $v(c_0, c_1, c_2, ...)$ with all the “usual properties”
  - Separable across time
  - Strictly increasing in each $c_i$ individually
  - Diminishing marginal utility in each of $c_i$ individually

- Expected lifetime discounted utility

$$E_0 v(c_0, c_1, c_2, 1, ...) = u(c_0) + \beta E_0 u(c_1) + \beta^2 E_0 u(c_2) + \beta^3 E_0 u(c_3) + ...$$

  “Planning time” aligned with “calendar time” by starting at period zero

- Conditional on information set of period zero

- First consider deterministic

$$\sum_{t=0}^T \beta^t u(c_t)$$

and then consider limit

$$\lim_{\beta \to 0} \sum_{t=0}^T \beta^t u(c_t)$$

before turning to stochastic

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**DETERMINISTIC – SEQUENTIAL ANALYSIS**

- Planning horizon $T$

$$\max_{\{c_t, a_t\}_{t=0}^T} \sum_{t=0}^T \beta^t u(c_t) \quad \text{subject to} \quad c_i + a_i = y_i + (1 + r_{i-1})a_{i-1}, \quad t = 0, 1, 2, ..., T$$

- FOCs

  - Euler equations

  $$u'(c_0) = \beta u'(c_1)(1 + r_0)$$
  $$u'(c_1) = \beta u'(c_2)(1 + r_1)$$
  $$u'(c_T) = \beta u'(c_{T-1})(1 + r_{T-1})$$

- Do we need any conditions at all about $a_T$?
DETERMINISTIC – SEQUENTIAL ANALYSIS

- Planning horizon \( T \)

\[
\max_{\{c_t, a_t\}_{t=0}^T} \sum_{t=0}^T \beta^t u(c_t) \quad \text{subject to} \quad c_t + a_t = y_t + (1 + r_{t-1})a_{t-1}, \quad t = 0, 1, 2, \ldots, T
\]

- Need to solve for \( 2(T+1) \) unknowns
  - \( c_0, c_1, c_2, \ldots, c_T \) \( T+1 \) unknowns
  - \( a_0, a_1, a_2, \ldots, a_T \) \( T+1 \) unknowns

- Need \( 2(T+1) \) conditions
  - Flow budget constraints for periods 0, 1, 2, \ldots, \( T \) \( T+1 \) conditions
  - Euler equations for periods 0, 1, 2, \ldots, \( T-1 \) \( T \) conditions
  - Require one more condition – a (second) boundary condition

BOUNDARY CONDITIONS

- \( a_{-1} \) provides initial condition on problem
- Requiring second boundary condition suggests problem is a (sequence of) second-order difference (differential) equations

- Consider Euler equation for period \( t \)

\[
u'(c_t) = \beta u'(c_{t-1})(1 + r_t)
\]

Substitute \( c_t \) and \( c_{t+1} \) using \( t \) and \( t+1 \) budget constraints

\[
u'(y_t + (1 + r_{t-1})a_{t-1} - a_t) = \beta u'(y_{t+1} + (1 + r_t)a_t - a_{t+1})(1 + r_t)
\]

- Second-order difference equation in \( a \)
  - Need two boundary conditions to solve

- Second boundary condition to require?
  - Candidate: \( a_s = \text{constant for } 0 \leq s \leq T \)
  - Arbitrary...
  - Does one arise naturally from the economics of the problem?
BOUNDARY CONDITIONS

- "No-Ponzi game" condition
  - Would be "optimal" to end with (infinite) debt
  - Ruling out strictly negative assets at end of planning horizon seems natural
  - So let's require $a_T \geq 0$

- No-Ponzi a "no-bankruptcies" restriction
  - Present in most models (whether aware of it or not...)
  - (NB: Models of "default" do exist...but often are "optimal defaults")
  - NOT quite the complete second boundary condition we need...

- $\beta_s \lambda_s$ measures (present-discounted) marginal value of resources in period $s$
  - Starting from allocation that solves problem

- Suppose $(c_0^*, c_1^*, c_2^*, \ldots , c_T^*; a_0^*, a_1^*, a_2^*, \ldots , a_T^*)$ solves problem
  - An interior allocation

- Let $V(.)$ be maximized value (utility)

$$V(a_{-1};) \equiv \sum_{t=0}^{T} \beta^t \left[ u(c_t^*) + \lambda_t \left[ y_t + (1+r_t)a_{t-1}^* - c_t^* - a_t^* \right] \right]$$

- Marginal injection of resources in period $s$ increases value of objective function by
  $$\frac{\partial V(a_{-1};)}{\partial y_s} = \beta^s \lambda_s$$

- Envelope Theorem – do not need to consider effect of marginal change in $y_s$ on optimal choices

- Suggests a constructive "proof" of a second boundary condition
CONSTRUCTING A BOUNDARY CONDITION

- **Define period-$$t$$ objective function**
  $$\tilde{u}(a_{t-1}, a_t) \equiv u(y_t + (1 + r_{t-1})a_{t-1} - a_t) \Rightarrow \tilde{u}_t(a_{t-1}, a_t) > 0, \tilde{u}_2(a_{t-1}, a_t) < 0$$

- **Dynamic optimization problem**
  $$\max_{\{a_t\}_{t=0}^T} \sum_{t=0}^T \beta^t \tilde{u}(a_{t-1}, a_t)$$

- **First-order necessary conditions**
  $$\tilde{u}_t(a_{t-1}^*, a_t^*) + \beta \tilde{u}_t(a_{t-1}^*, a_{t+1}^*) = 0, \forall t = 0, 1, 2, ..., T - 1$$

- **Define maximized value starting from date $$t$$**
  $$\tilde{V}(a_{t-1}^*, a_t^*) = \sum_{t=0}^T \beta^{t-t} \tilde{u}(a_{t-1}^*, a_t^*)$$

**Assumptions**

- **Optimal solution is interior (i.e., $$c_t > 0 \forall t$$)**
  - (weak assumption)

- **$$\tilde{u}(0, 0) \geq 0$$**
  - (weak assumption)

- **Implies $$\tilde{V}(0) \geq 0$$**

- **Lifetime utility is finite for all feasible paths**
  - (weak assumption)

- **Implies $$\lim_{T \to \infty} \beta^T \tilde{V}(a_{t-1}^*, a_t^*) = 0$$**

- **Also note $$\tilde{V}(\cdot)$$**
  - Is increasing
  - Is concave because $$\tilde{u}(\cdot)$$ is concave
**CONSTRUCTING A BOUNDARY CONDITION**

- **Envelope condition**
  \[
  \frac{\partial \tilde{V}(a_{t,i}^*, a_t^*)}{\partial a_{t,i}} \equiv \tilde{u}_t(a_{t,i}^*, a_t^*)
  \]

- **First-order necessary conditions**
  \[
  \tilde{u}_t(a_{t-1,i}^*, a_t^*) + \beta \tilde{u}_t(a_t^*, a_{t+1}^*) = 0, \quad \forall t = 0, 1, 2, ..., T - 1
  \]

  \[\Rightarrow -\tilde{u}_t(a_{t-1,i}^*, a_t^*) \cdot a_t^* = \beta \tilde{u}_t(a_t^*, a_{t+1}^*) \cdot a_t^* = \beta \frac{\partial \tilde{V}(a_t^*, \ldots)}{\partial a_t^*} \cdot a_t^* \leq \beta \tilde{V}(a_t^*, \ldots)\]

  Recall < 0  
  By envelope condition  
  Because \(V(0) \geq 0\) and \(V(.)\) concave

  > 0  
  By No-Ponzi condition, \(a_T^* \geq 0\) at end of planning horizon

- **Implies**
  \[\beta^{T+1} \tilde{V}(a_T^*, \ldots) \geq -\beta^T \tilde{u}_T(a_{T-1,i}^*, a_T^*) a_T^* \geq 0\]

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**CONSTRUCTING A BOUNDARY CONDITION**

\[\beta^{T+1} \tilde{V}(a_T^*, \ldots) \geq -\beta^T \tilde{u}_T(a_{T-1,i}^*, a_T^*) a_T^* \geq 0\]

- **Consider** \(T \rightarrow \infty\)
  
  \[\lim_{T \rightarrow \infty} \beta^{T+1} \tilde{V}(a_T^*, \ldots) \geq \lim_{T \rightarrow \infty} -\beta^T \tilde{u}_T(a_{T-1,i}^*, a_T^*) a_T^* \geq 0\]

  \[= \beta^T u^*(c_T^*) a_T^*\]

  \[\Rightarrow \lim_{T \rightarrow \infty} \beta^{T+1} \tilde{V}(a_T^*, \ldots) \geq \lim_{T \rightarrow \infty} \beta^T u^*(c_T^*) a_T^* \geq 0\]

  \[= 0\] because lifetime utility assumed finite for any feasible path
BOUNDARY CONDITIONS

- "No-Ponzi game" condition
  - Would be "optimal" to end with (infinite) debt
  - Ruling out strictly negative assets at end of planning horizon seems natural
  - So let’s require $a_T \geq 0$

- “Transversality condition”
  \[
  \lim_{T \to \infty} \beta^T u'(c_T^*) a_T^* = 0
  \]
  - The second boundary condition (along with initial condition)
  - Suboptimal to end planning horizon with positive assets that would have strictly raised lifetime discounted utility
  - Sufficiency straightforward to establish
  - Necessity much more difficult (Kamihigashi, 2001 *Econometrica*)

- TVC is stronger statement than No-Ponzi condition

DETERMINISTIC – SEQUENTIAL ANALYSIS

- Complete solution of $T$-period planning horizon?

- Solution to consumer problem is a consumption and asset sequence $\{c_t^*, a_t^*\}_{t=0}^{T}$ that satisfies
  - Sequence of flow budget constraints
    \[ c_t^* + a_t^* = y_t + (1 + r_{t-1})a_{t-1}^*, \quad t = 0, 1, 2, \ldots, T \]
  - Sequence of Euler equations
    \[ u'(c_t^*) = \beta u'(c_{t+1}^*) (1 + r_t), \quad t = 0, 1, 2, \ldots, T - 1 \]
  - "TVC" for finite horizon
    \[ \beta^T u'(c_T^*) a_T^* = 0 \iff a_T^* = 0 \]
  - Taking as given \( \{r_t, y_t\}_{t=0}^{T}, \{a_t, c_t\} \)

Exactly as in two-period model!
Complete solution of infinite-period horizon?

Solution to consumer problem is a consumption and asset sequence \( \{c_t^*, a_t^*\}_{t=0}^{\infty} \) that satisfies

- Sequence of flow budget constraints:
  \[ c_t^* + a_t^* = y_t + (1 + r_{t-1})a_{t-1}^*, \quad t = 0, 1, 2, \ldots \]

- Sequence of Euler equations:
  \[ u'(c_t^*) = \beta u'(c_{t+1}^*)(1 + r_t), \quad t = 0, 1, 2, \ldots \]

- TVC
  \[ \lim_{t \to \infty} \beta'u'(c_t^*)a_t^* = 0 \]

taking as given \( \{r_t, y_t\}_{t=0}^{\infty}, a_{-1}, r_{-1} \)