INTERTEMPORAL MODELS: BASICS OF DYNAMIC PROGRAMMING

SEPTEMBER 26, 2011

DYNAMIC PROGRAMMING

- **Can we represent intertemporal problems recursively?**
  - **Rather than sequentially**
- **Benefits**
  - **Allows application of series of theorems/results that guarantee a solution exists in the space of functions**
  - **Allows application of series of theorems/results that help find solution in the space of functions**
  - **Computational algorithms require it – computers can't handle infinite-dimensional objects!**
- **Costs**
  - **May rule out some solutions to the original (sequential) problem**
  - **Requires (a lot?) more structure on the problem**
  - **Sometimes (often?) not obvious how to recast sequential problem as recursive problem**
- **Ljungqvist and Sargent (2004, p. 16)**

  "The art in applying recursive methods is to find a convenient definition of a state. It is often not obvious what the state is, or even whether a finite-dimensional state exists."

Dynamic Programming

- Can we represent intertemporal problems recursively?
  - Rather than sequentially
- Benefits
  - Allows application of series of theorems/results that guarantee a solution exists in the space of functions
  - Allows application of series of theorems/results that help find solution in the space of functions
  - Computational algorithms require it – computers can't handle infinite-dimensional objects!
- Costs
  - May rule out some solutions to the original (sequential) problem
  - Requires (a lot?) more structure on the problem
  - Sometimes (often?) not obvious how to recast sequential problem as recursive problem
- Start with deterministic case
  - (Fairly) straightforward
  - Stochastic case requires more structure

From Sequential to Recursive

- Lagrangian of consumer problem, with planning horizon T

$$V^0(a_{-1}, r_0; .) \equiv \max_{\{c_t, a_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t \left[u(c_t) + \lambda_t \left(y_t + (1+r_{t-1})a_{t-1} - c_t - a_t \right) \right]$$

- State variables of consumer problem at beginning of any period s
  - a_{s-1} (accumulation variable) – the critical one b/c a_τ, $\tau \geq s$, are choices
  - r_s (price-taker)
  - A sufficient summary of the dynamic position of the environment in which the consumer operates

FROM SEQUENTIAL TO RECURSIVE

- **Lagrangian of consumer problem, with planning horizon *T***

$$V^0(a_{-1}, r_0;.) \equiv \max_{\{c_t, a_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t \left[u(c_t) + \lambda_t \left(y_t + (1+r_{t-1})a_{t-1} - c_t - a_t \right) \right]$$

- **State variables of consumer problem at beginning of any period *s***
  - a_{s-1} **(accumulation variable) – the critical one b/c a_τ, $\tau \geq s$, are choices**
  - r_s **(price-taker)**
  - **A sufficient summary of the dynamic position of the environment in which the consumer operates**

- **Define $V^0(a_{-1}, r_0;.)$ as value function starting from period zero**
  - **The maximized value of the constrained optimization problem**
  - **As function of period-zero parameters of the problem**

- **Goal: recast problem of finding optimal sequence $\{c_t, a_t\}_{t=0,1,2,\ldots T}$ into problem of finding functions $\{V^t(.)\}_{t=0,1,2,\ldots T}$**
  - **(Actually, find $V^t(.)$ along with two other functions)**

FROM SEQUENTIAL TO RECURSIVE

- **Write out more explicitly**

$$V^0(a_{-1}, r_0;.) \equiv \max_{\{c_0, a_0, c_t, a_t\}_{t=1}^{\infty}} \left\{ \begin{array}{l} u(c_0) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c_0 - a_0 \right) \\ \qquad + \sum_{t=1}^{T} \beta^t \left[u(c_t) + \lambda_t \left(y_t + (1+r_{t-1})a_{t-1} - c_t - a_t \right) \right] \end{array} \right\}$$

FROM SEQUENTIAL TO RECURSIVE

- Write out more explicitly

$$V^0(a_{-1},r_0;.) \equiv \max_{\{c_0,a_0,c_t,a_t\}_{t=1}^{\infty}} \left\{ \begin{aligned} & u(c_0)+\lambda_0\left(y_0+(1+r_{-1})a_{-1}-c_0-a_0\right) \\ & \quad + \sum_{t=1}^{T} \beta^t \left[u(c_t)+\lambda_t\left(y_t+(1+r_{t-1})a_{t-1}-c_t-a_t\right)\right] \end{aligned} \right\}$$

↓ Separate terms

$$V^0(a_{-1},r_0;.) \equiv \max_{c_0,a_0} \left\{ u(c_0)+\lambda_0\left(y_0+(1+r_{-1})a_{-1}-c_0-a_0\right)\right\}$$

$$+\max_{c_0,a_0}\left\{ \max_{\{c_t,a_t\}_{t=1}^{\infty}} \sum_{t=1}^{T} \beta^t \left[u(c_t)+\lambda_t\left(y_t+(1+r_{t-1})a_{t-1}-c_t-a_t\right)\right]\right\}$$

Note the max inside the max

↓ Adjust β factors

FROM SEQUENTIAL TO RECURSIVE

Adjust β factors →

$$V^0(a_{-1},r_0;.) \equiv \max_{c_0,a_0} \left\{ u(c_0)+\lambda_0\left(y_0+(1+r_{-1})a_{-1}-c_0-a_0\right)\right\}$$

$$+\beta\cdot\max_{c_0,a_0}\underbrace{\left\{ \max_{\{c_t,a_t\}_{t=1}^{\infty}} \sum_{t=1}^{T} \beta^{t-1} \left[u(c_t)+\lambda_t\left(y_t+(1+r_{t-1})a_{t-1}-c_t-a_t\right)\right]\right\}}$$

FROM SEQUENTIAL TO RECURSIVE

$$V^0(a_{-1}, r_0; .) \equiv \max_{c_0, a_0} \{u(c_0) + \lambda_0 (y_0 + (1+r_{-1})a_{-1} - c_0 - a_0)\}$$

Adjust β factors →

$$+\beta \cdot \max_{c_0, a_0} \left\{ \max_{\{c_t, a_t\}_{t=1}^{\infty}} \sum_{t=1}^{T} \beta^{t-1} \left[u(c_t) + \lambda_t (y_t + (1+r_{t-1})a_{t-1} - c_t - a_t) \right] \right\}$$

$V^0(a_{-1}, r_0; .)$ is value function starting from period 0.

<u>Bellman Principle of Optimality:</u> optimal decisions in the initial period induce a future state, from which (future) decisions are optimal (Bellman, 1957)

$\equiv V^1(a_0, r_1; .)$, value function starting from period 1.

The value resulting from optimal decisions starting from period 1.

FROM SEQUENTIAL TO RECURSIVE

$$V^0(a_{-1}, r_0; .) \equiv \max_{c_0, a_0} \{u(c_0) + \lambda_0 (y_0 + (1+r_{-1})a_{-1} - c_0 - a_0)\}$$

Adjust β factors →

$$+\beta \cdot \max_{c_0, a_0} \left\{ \max_{\{c_t, a_t\}_{t=1}^{\infty}} \sum_{t=1}^{T} \beta^{t-1} \left[u(c_t) + \lambda_t (y_t + (1+r_{t-1})a_{t-1} - c_t - a_t) \right] \right\}$$

$V^0(a_{-1}, r_0; .)$ is value function starting from period 0.

<u>Bellman Principle of Optimality:</u> optimal decisions in the initial period induce a future state, from which (future) decisions are optimal (Bellman, 1957)

$\equiv V^1(a_0, r_1; .)$, value function starting from period 1.

The value resulting from optimal decisions starting from period 1.

Recursive representation of consumer problem

$$V^0(a_{-1}, r_0; .) \equiv \max_{c_0, a_0} \{u(c_0) + \lambda_0 (y_0 + (1+r_{-1})a_{-1} - c_0 - a_0) + \beta \cdot V^1(a_0, r_1; .)\}$$

- **Bellman Equation**
- **Can analyze optimization problem for period zero…**
  - **…given Bellman Principle of Optimality holds**
  - **(But how do $V^0(.)$ and $V^1(.)$ relate to each other?)**

BELLMAN EQUATION

- **Bellman Equation**

$$V^0(a_{-1},r_0;.) \equiv \max_{c_0,a_0}\left\{u(c_0)+\lambda_0\left(y_0+(1+r_{-1})a_{-1}-c_0-a_0\right)+\beta\cdot V^1(a_0,r_1;.)\right\}$$

  - Starting point for recursive analysis
  - Applicable to finite *T*-period or $T\to\infty$ problems
  - Construction requires identifying **state variables** of optimization problem

- ***T*-period problem**
  - Solution involves sequence of functions $V^0(.)$, $V^1(.)$, … , $V^{T-1}(.)$, $V^T(.)$
  - $V^i(.)$ functions in general will differ – reflecting time until end of planning horizon
  - E.g., maximized value starting from age = 60 different from maximized value starting from age = 30 (intuitively)

- **Infinite-horizon problem**
  - Deterministic case: $V(.) \equiv V^i(.) = V^j(.)\ \forall\ i,j$
  - Always an infinity of periods left to go

Stochastic case?

Requires more structure…

BELLMAN EQUATION

- **Bellman Equation (for $T\to\infty$)**

$$V(a_{-1},r_0;.) \equiv \max_{c_0,a_0}\left\{u(c_0)+\lambda_0\left(y_0+(1+r_{-1})a_{-1}-c_0-a_0\right)+\beta\cdot V(a_0,r_1;.)\right\}$$

- Use to characterize optimal decisions
- Period-0 FOCs

c_0: $u'(c_0)-\lambda_0=0$

a_0: $-\lambda_0+\beta V_1(a_0,r_1;.)=0$

How to compute $V_1(.)$?

BELLMAN EQUATION

- **Bellman Equation (for $T \to \infty$)**

$$V(a_{-1}, r_0;.) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c_0 - a_0 \right) + \beta \cdot V(a_0, r_1;.) \right\}$$

- **Use to characterize optimal decisions**
- **Period-0 FOCs**

c_0: $u'(c_0) - \lambda_0 = 0$

a_0: $-\lambda_0 + \beta V_1(a_0, r_1;.) = 0$ **How to compute $V_1(.)$?**

Return to this …

- ***Suppose*** **optimal choice characterized by $c_0 = c(a_{-1};.)$, $a_0 = a(a_{-1};.)$ ($c(.)$ and $a(.)$ time-invariant functions in infinite-period problem)**
  - **Insert in value function (can now drop max operator)**

$$V(a_{-1}, r_0;.) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1;.)$$

  - **Now compute marginal**

Deterministic Dynamic Programming

BELLMAN EQUATION

- **Bellman Equation (for $T \to \infty$)**

$$V(a_{-1}, r_0;.) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1;.)$$

  - **Now compute marginal (suppress r argument of $c(.)$ and $a(.)$ functions)**

$V_1(a_{-1}, r_0;.) =$

BELLMAN EQUATION

- **Bellman Equation (for $T \to \infty$)**

$$V(a_{-1}, r_0; .) \equiv u(c(a_{-1})) + \lambda_0\left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1})\right) + \beta \cdot V(a(a_{-1}), r_1; .)$$

- **Now compute marginal (suppress *r* argument of *c*(.) and *a*(.) functions)**

$$V_1(a_{-1}, r_0; .) =$$

$$\Rightarrow \quad V_1(a_{-1}, r_0; .) =$$

Envelope Condition

- **Envelope Theorem**
  - **In computing first-order effects of changes in a problem's parameters on the maximized value, can ignore how optimal choices will adjust**
    - **Intuition: because already at a max (marginal costs = marginal benefits)**
  - **Need only consider the direct effect**

Note: envelope theorem has nothing to do with dynamic programming

BELLMAN EQUATION

- **Bellman Equation (for $T \to \infty$)**

$$V(a_{-1}, r_0; .) \equiv u(c(a_{-1})) + \lambda_0\left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1})\right) + \beta \cdot V(a(a_{-1}), r_1; .)$$

- **Use to characterize optimal decisions**

- **Period-0 FOCs, now evaluated using $c(a_{-1})$, $a(a_{-1})$**

***c*₀**: $u'(c(a_{-1})) - \lambda_0 = 0$

***a*₀**: $-\lambda_0 + \beta V_1(a_0(a_{-1}), r_1; .) = 0$

Env: $V_1(a(a_{-1}), r_1; .) = \lambda_1(1+r_0)$

$$\Rightarrow \quad u'(c(a_{-1})) = \beta(1+r_0)u'(c(a_0))$$

- **Seems like usual Euler equation from sequential analysis (deterministic)…**

Deterministic – Recursive Analysis

- Solution of infinite-horizon consumer problem (starting from date zero)…
- …is a consumption decision rule $c(a_{-1};.)$, asset decision rule $a(a_{-1};.)$, and value function $V(a_{-1};.)$ that satisfies
  - Bellman equation

$$V(a_{-1},r_0;.) \equiv u(c(a_{-1})) + \lambda_0\left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1})\right) + \beta \cdot V(a(a_{-1}),r_1;.)$$

  - Euler equation

$$u'(c(a_{-1})) = \beta V_1(a(a_{-1}),r_1;.) \quad \overset{\text{by envelope theorem}}{\longleftrightarrow} \quad u'(c(a_{-1})) = \beta(1+r_0)u'(c(a_{-1}))$$

    - which is the TVC in the limit $t \to \infty$: $\lim_{t\to\infty} \beta^t u'(c(a_{t-1}^*)) \cdot a(a_{t-1}^*) = 0$
  - Budget constraint

$$y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) = 0$$

taking as given $\left(a_{-1}, r_0, r_{-1}\right)$

Deterministic – Sequential Analysis

- Solution of infinite-horizon consumer problem (starting from date zero)…
- is a consumption and asset sequence $\left\{c_t^*, a_t^*\right\}_{t=0}^{\infty}$ that satisfies
  - Sequence of flow budget constraints

$$c_t^* + a_t^* = y_t + (1+r_{t-1})a_{t-1}^*, \quad t = 0,1,2,\ldots$$

  - Sequence of Euler equations

$$u'(c_t^*) = \beta u'(c_{t+1}^*)(1+r_t), \quad t = 0,1,2,\ldots$$

    - which is the TVC in the limit $t \to \infty$: $\lim_{t\to\infty} \beta^t u'(c_t^*)a_t^* = 0$

taking as given $\left(\left\{r_t, y_t\right\}_{t=0}^{\infty}, a_{-1}, r_{-1}\right)$

Does solution to recursive problem coincide with solution to sequential problem?

RECURSIVE VS. SEQUENTIAL ANALYSIS

- Does solution to recursive problem coincide with solution to sequential problem?
- Does solution to sequential problem coincide with solution to recursive problem?

RECURSIVE VS. SEQUENTIAL ANALYSIS

- Does solution to recursive problem coincide with solution to sequential problem?
- Does solution to sequential problem coincide with solution to recursive problem?

- In constructing Bellman representation ($T \to \infty$ case), the imposition of time-invariant functions $c(a)$, $a(a)$ potentially limits the class of solutions
  - In original sequential formulation, this is neither explicitly nor implicitly a requirement of the solution!
- In general (here without proof…)
  - Solution to the sequential problem is also a solution to the recursive problem
  - Solution to the recursive problem is also a solution to the sequential problem provided some further regularity conditions hold
- Stokey, Lucas, Prescott text (1989)

RECURSIVE VS. SEQUENTIAL ANALYSIS

- **So why go recursive?**
  - **Allows application of series of theorems/results that guarantee a solution exists in the space of functions**
  - **Allows application of series of theorems/results that help find solution in the space of functions**

Underlying theory:
Contraction Mapping Theorem, Blackwell's Sufficient Conditions for a Contraction, Theorem of the Maximum

  - **Computational algorithms require it – computers can't handle infinite-dimensional objects!**
    - **Econ 701, 702, 630: various computational algorithms**

Dynamic Programming

RECURSIVE VS. SEQUENTIAL ANALYSIS

- **So why go recursive?**
  - **Allows application of series of theorems/results that guarantee a solution exists in the space of functions**
  - **Allows application of series of theorems/results that help find solution in the space of functions**

Underlying theory:
Contraction Mapping Theorem, Blackwell's Sufficient Conditions for a Contraction, Theorem of the Maximum

  - **Computational algorithms require it – computers can't handle infinite-dimensional objects!**
    - **Econ 701, 702, 630: various computational algorithms**

- **Can't really "choose" whether want to analyze problem sequentially or recursively**
  - **All but the most limited of problems/models require computational solution**
  - **In which case model analysis is recursive**

- **What about stochastic dynamic programming?**
  - **Even more structure required….**

STOCHASTIC DYNAMIC PROGRAMMING

- Even more structure required on the problem to recursively solve dynamic stochastic optimization problems

- Main (new) technical problem
  - Branching of event tree at each of T periods (possibly $T \to \infty$)

- Main technical solution / assumption
  - Assume risk follows Markov process
  - Which enables series of theorems / results from deterministic dynamic programming to work in stochastic case…
  - …given further technical regularity assumptions (Econ 602)

- Illustrate the technical problem by building on the stochastic two-period model