BASICS OF DYNAMIC PROGRAMMING (CONTINUED)

SEPTEMBER 28, 2011

RECURSIVE REPRESENTATION

- **State variables**
  - **A sufficient summary, as of the start of period *t*, of the dynamic position of the environment in which the maximizing agent operates**
  - **"Environment" of the agent – what needs to be known in order to optimize in period *t*?**
    - **Individual-specific quantities**
    - **Market prices**
    - **Government policies**
    - **(Fixed structural parameters – will omit from state vector for parsimony)**

The usual suspects

Important: states can be endogenous or exogenous

- **Ljungqvist and Sargent (2004, p. 16)**

  "The art in applying recursive methods is to find a convenient definition of a state. It is often not obvious what the state is, or even whether a finite-dimensional state exists."

Recursive Representation

- **State variables**
  - A sufficient summary, as of the start of period *t*, of the dynamic position of the environment in which the maximizing agent operates
  - "Environment" of the agent – what needs to be known in order to optimize in period *t*?

    The usual suspects
    - Individual-specific quantities
    - Market prices
    - Government policies

    Important: states can be endogenous or exogenous
    - (Fixed structural parameters – will omit from state vector for parsimony)
  - "Sufficient" – there are no other objects (quantities, prices, govt policies, etc.) that must be known in order to optimize in period *t*
  - Concept well-defined for both finite-*T* and $T \to \infty$ problems
  - KEY: Period-*t* decisions are function of the period-*t* state variables
- **Ljungqvist and Sargent (2004, p. 16)**

  "The art in applying recursive methods is to find a convenient definition of a state. It is often not obvious what the state is, or even whether a finite-dimensional state exists."

Bellman Equation

- **Bellman Equation**

$$V^0(a_{-1}, r_0; .) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c_0 - a_0 \right) + \beta \cdot V^1(a_0, r_1; .) \right\}$$

  - Starting point for recursive analysis
  - Applicable to finite *T*-period or $T \to \infty$ problems
  - Construction requires identifying state variables
- ***T*-period problem**
  - Solution involves sequence of functions $V^0(.)$, $V^1(.)$, … , $V^{T-1}(.)$, $V^T(.)$
  - $V^i(.)$ functions in general will differ – reflecting time until end of planning horizon
  - E.g., maximized value starting from age = 60 different from maximized value starting from age = 30 (intuitively)
- **Infinite-horizon problem ("stationary" environment)**
  - Deterministic case: $V(.) \equiv V^i(.) = V^j(.)\ \forall\, i,j$
  - Always an infinity of periods left to go

BELLMAN EQUATION

- **Bellman Equation (for $T \to \infty$)**

$$V(a_{-1}, r_0; .) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c_0 - a_0 \right) + \beta \cdot V(a_0, r_1; .) \right\}$$

- **Use to characterize optimal decisions**

- **Period-0 FOCs, evaluated using time-invariant $c(a_{-1})$, $a(a_{-1})$**

$$\left.\begin{array}{ll} \mathbf{c_0}: & u'(c(a_{-1})) - \lambda_0 = 0 \\ \mathbf{a_0}: & -\lambda_0 + \beta V_1(a_0(a_{-1}), r_1; .) = 0 \\ \textbf{Env}: & V_1(a(a_{-1}), r_1; .) = \lambda_1(1+r_0) \end{array}\right\} \quad u'(c(a_{-1})) = \beta(1+r_0)u'(c(a_0))$$

- **Seems like usual Euler equation from sequential analysis (deterministic)…**

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$$V(a_{-1}, r_0; .) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1; .)$$

- **Seems like a two-period problem**
  - **In terms of (value) functions, not in terms of choice variables**
  - **Optimize in current period**
  - **Optimize next period (Bellman's Principle of Optimality)**

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  - **Optimize in current period**
  - **Optimize next period (Bellman's Principle of Optimality)**
- **Common notation**
  - **Use *x* for current-period variables**
  - **Use *x'* for next-period variables**
- **Bellman Equation**

$$V(a,r;.) \equiv u(\underbrace{c(a)}_{=\,c}) + \lambda \left(y + (1+r_{-1})a - \underbrace{c(a)}_{=\,c} - \underbrace{a(a)}_{=\,a'} \right) + \beta \cdot V(\underbrace{a(a)}_{=\,a'}, r';.)$$

- **Euler equation**

$$u'(\overbrace{c(a)}^{=\,c}) = \beta(1+r)u'(\overbrace{c(a')}^{=\,c'})$$

RECURSIVE VS. SEQUENTIAL ANALYSIS

- **So why go recursive?**
  - **Allows application of series of theorems/results that guarantee a solution exists in the space of functions**
  - **Allows application of series of theorems/results that help find solution in the space of functions**

Underlying Theory:
Contraction Mapping Theorem, Blackwell's Sufficient Conditions for a Contraction, Theorem of the Maximum

- **Suppose *V*(.) exists**
- **Procedure for finding *V*(.) and associated decision rules: iterate on Bellman Equation starting from any arbitrary initial guess – call it V^1(.)**

initial guess (some parametric form)

$$V(a,r;.) \equiv \max_{c,a'} \left\{ u(c) + \lambda \left(y + (1+r)a - c - a' \right) + \beta \cdot V^1(a', r';.) \right\}$$

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  - Conduct maximization
    - Gives functions $c(a)$ and $a(a)$
    - These are candidate (optimal) decision rules
  - Insert candidate $c(a)$ and $a(a)$ into RHS of Bellman Equation – generates $V^2(.)$

Does $V^2(.) = V^1(.)$?

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  - Conduct maximization
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  - Insert candidate $c(a)$ and $a(a)$ into RHS of Bellman Equation – generates $V^2(.)$

If no, insert $V^2(.)$ on RHS and repeat

Does $V^2(.) = V^1(.)$? If yes, stop. Have found $V(.)$ (= $V^2(.)$ = $V^1(.)$)

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e.g., value function iteration

  - Computational algorithms require it – computers can't handle infinite-dimensional objects!
    - Econ 701, 702, 630: various computational algorithms
- Can't "choose" whether to analyze problem sequentially or recursively
  - All but the most limited of problems require computational solution
  - In which case model analysis is recursive

Dynamic Programming

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- "Solving model sequentially"
  - Doesn't seem recursive… $\longrightarrow$ $u(c_t) = \beta(1+r_t)u'(c_{t+1})$

Imposing recursivity on solution

  - …but computational implementation requires time-invariant decision rule $u(c(a_{t-1})) = \beta(1+r_t)u'(c(a_t))$

Recursive vs. Sequential Analysis

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- Can't "choose" whether to analyze problem sequentially or recursively
  - All but the most limited of problems require computational solution
  - In which case model analysis is recursive
- What about stochastic dynamic programming?
  - Even more structure required....
  - The key assumption is Markov risk

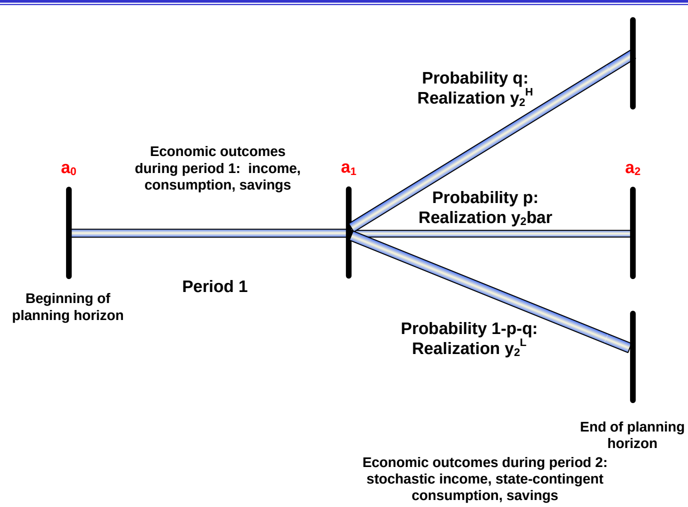
Intertemporal Models: Stochastic Dynamic Programming

September 28, 2011

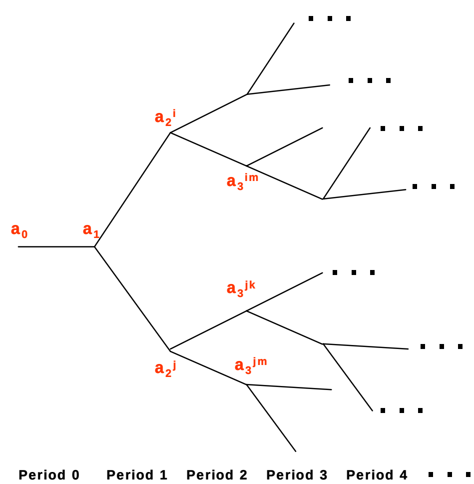
STOCHASTIC DYNAMIC PROGRAMMING

- Even more structure required on the problem to recursively solve dynamic stochastic optimization problems
- Main (new) technical problem
  - Branching of event tree at each of T periods (possibly $T \to \infty$)
- Main technical solution / assumption
  - Assume risk follows Markov process
  - Which enables series of theorems / results from deterministic dynamic programming to work in stochastic case…
  - …given further technical regularity assumptions (Econ 602)
- Illustrate technical problem by extending stochastic two-period model

EVENT TREE



Event Tree



- In general could be any arbitrary unfolding of exogenous risk
  - e.g., if state *i* realized in period 16, then 14 possible states in period 17; but if state *j* realized in period 16, then 8 possible states in period 17

    OR

  - e.g., probability of state *i* in period *t* depends on event in period *t*-100000
- Number of decision rules to solve explodes
- Intractable!!!
- "Curse of dimensionality"
- Requires a lot of structure on exogenous risk

Risk Structure

Assumptions

- Set of realizations of exogenous state variable is independent of date

$$S2 = S3 = S4 = S5 = \ldots. = ST\text{-}1 = ST$$

RISK STRUCTURE

Assumptions

- **Set of realizations of exogenous state variable is independent of date**
- **Probability of realization of exogenous state variable in period *t* depends only on outcomes in period *t*-1**
  - **Suppose X_t is a stochastic process and x_t is a particular realization**
  - **X_t is a Markov process if**

$$\Pr\left(X_t = x_t \mid X_{t-1} = x_{t-1}, X_{t-2} = x_{t-2}, X_{t-3} = x_{t-3}, \ldots, X_{t-10000} = x_{t-10000}, \ldots\right)$$

$$= \Pr\left(X_t = x_t \mid X_{t-1} = x_{t-1}\right)$$ **CONDITIONAL probability depends on only *t*-1**

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$$= \Pr\left(X_t = x_t \mid X_{t-1} = x_{t-1}\right)$$ **CONDITIONAL probability depends on only *t*-1**

  - **Not as restrictive as it may seem – could have finite lags in process**
  - **E.g.**
  - **Just can't have infinite lags (in principle) or "too many" (finite) lags (in computational practice)**

RISK STRUCTURE

Assumptions

- Set of realizations of exogenous state variable is independent of date
- Probability of realization of exogenous state variable in period t depends only on outcomes in period t-1
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- Exogenous state variable is Markov process + assumption/result that decision rules are time-invariant (for $T \to \infty$) functions of state variables

$\Rightarrow$ **Endogenous processes are Markov**
given several regularity assumptions

Underlying theory:
Stokey, Lucas, Prescott (1989, Chapters 8-12)

STOCHASTIC – SEQUENTIAL ANALYSIS

- Planning horizon $T \to \infty$
- Exogenous state drawn from set S (could be continuous or discrete)
- Suppose single asset with state-contingent r (will illustrate main ideas)

$$\max_{\{c_t, a_t\}_{t=0}^{T}} E_0 \sum_{t=0}^{T} \beta^t u(c_t) \text{ subject to } \begin{cases} c_t + a_t = y_t + (1+r_t)a_{t-1}, \quad t = 0,1,2,\ldots T \\ \text{state-contingent budget constraints in } t > 0 \end{cases}$$

- FOCs

c_0:

a_0:

c_1:

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- **FOCs**

c_0: $u'(c_0) - \lambda_0 = 0$

a_0: $-\lambda_0 + \beta E_0\left[\lambda_1(1+r_1)\right] = 0$ → Just as in stochastic two-period model → $1 = E_0\left[\dfrac{\beta\lambda_1}{\lambda_0}(1+r_1)\right]$

c_1: $\beta E_0 u'(c_1) - \beta E_0 \lambda_1 = 0$ → Holds for each state → $E_0 u'(c_1^j) = E_0 \lambda_1^j, \ \forall j \in S$

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a_1: $-\beta E_0 \lambda_1 + \beta^2 E_0\left[\lambda_2(1+r_2)\right] = 0$

c_2: $\beta^2 E_0 u'(c_2) - \beta^2 E_0 \lambda_2 = 0$ → Holds for each state → $E_0 u'(c_2^j) = E_0 \lambda_2^j, \ \forall j \in S$

Model Analysis

Stochastic – Markov Solution

- $\{X_t\}_{t=0,1,2,\ldots}$ is Markov process (exogenous and endogenous states)
  - Nothing about the probability distribution of X_{t+2} is known in period t that is not known in period $t+1$
    - Information set of period $t+1$ is superset of information set of period t
- Allows applying a law of iterated expectations
  - $E_t X_{t+2} = E_t [E_{t+1} X_{t+2}]$

$$E_0\lambda_1 = \beta E_0\left[\lambda_2(1+r_2)\right] \longrightarrow E_0\lambda_1 = \beta E_0\left[E_1\left(\lambda_2(1+r_2)\right)\right]$$

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- Date- and state-contingent decisions: decisions governed by this Euler condition are conditional on information set of period 1 (i.e., recursivity)

$$\longrightarrow E_1\lambda_1 = \beta E_1\left[\lambda_2(1+r_2)\right] \longrightarrow \lambda_1 = \beta E_1\left[\lambda_2(1+r_2)\right]$$

STOCHASTIC – SEQUENTIAL ANALYSIS

- **Planning horizon $T \to \infty$**
- **Exogenous state drawn from set S (could be continuous or discrete)**
- **Suppose single asset with state-contingent r (will illustrate main ideas)**

$$\max_{\{c_t, a_t\}_{t=0}^{T}} E_0 \sum_{t=0}^{T} \beta^t u(c_t) \quad \textbf{subject to} \quad \begin{cases} c_t + a_t = y_t + (1+r_t)a_{t-1}, \quad t = 0,1,2,\ldots T \\ \text{state-contingent budget constraints in } t > 0 \end{cases}$$

- **FOCs**

c_1: $\beta E_0 u'(c_1) - \beta E_0 \lambda_1 = 0$ → Holds for each state → $E_1 u'(c_1^j) = E_1 \lambda_1^j, \ \forall j \in S$

a_1: $-\beta E_0 \lambda_1 + \beta^2 E_0 \left[\lambda_2 (1+r_2)\right] = 0$ → Because Markov and state- and date-contingent decisions → $\lambda_1 = \beta E_1 \left[\lambda_2 (1+r_2)\right]$

c_2: $\beta^2 E_0 u'(c_2) - \beta^2 E_0 \lambda_2 = 0$ → Holds for each state → $E_2 u'(c_2^j) = E_2 \lambda_2^j, \ \forall j \in S$

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$$\max_{\{c_t, a_t\}_{t=0}^{T}} E_0 \sum_{t=0}^{T} \beta^t u(c_t) \quad \textbf{subject to} \quad \begin{cases} c_t + a_t = y_t + (1+r_t)a_{t-1}, \quad t = 0,1,2,\ldots T \\ \text{with uncertain realizations in } t > 0 \end{cases}$$

- **FOCs**

c_t: $\beta^t E_0 u'(c_t) - \beta^t E_0 \lambda_t = 0$ → Holds for each date and state → $u'(c_1^j) = \lambda_1^j, \ \forall j \in S$

a_t: $-\beta^t E_0 \lambda_t + \beta^{t+1} E_0 \left[\lambda_{t+1} (1+r_{t+1})\right] = 0$ → Because Markov and state- and date-contingent decisions → $\lambda_t = \beta E_t \left[\lambda_{t+1} (1+r_{t+1})\right]$

c_{t+1}: $\beta^{t+1} E_0 u'(c_{t+1}) - \beta^{t+1} E_0 \lambda_{t+1} = 0$ → Holds for each date and state → $u'(c_2^j) = \lambda_2^j, \ \forall j \in S$

- **One-period-ahead conditional expectation governs stochastic Euler condition**

STOCHASTIC – MARKOV SOLUTION

- Denote exogenous state variables as z (e.g., $z_t = [y_t, r_t]$)
- Solution of infinite-horizon consumer problem is a consumption decision rule $c(a, z;.)$, asset decision rule $a(a, z;.)$, and value function $V(a, z;.)$ that satisfies

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- Solution of infinite-horizon consumer problem is a consumption decision rule $c(a, z;.)$, asset decision rule $a(a, z;.)$, and value function $V(a, z;.)$ that satisfies
  - (Stochastic) Euler equation

$$u'(c(a,z)) = \beta E\left[u'(c(a',z'))(1+r')\right]$$

    - which is the (expectational) TVC in the limit $t \to \infty$:

$$\lim_{t\to\infty} E_0 \beta^t u'(c(a,z)) \cdot a(a,z) = 0$$

  - Budget constraint

$$y + (1+r)a - c(a,z) - a(a,z) = 0$$

Expectation in Bellman Equation

Transition from $z \to z'$

  - Bellman Equation

$$V(a,z;.) \equiv u(c(a,z)) + \lambda\left(y + (1+r)a - c(a,z) - a(a,z)\right) + \beta \cdot EV(a(a,z), z(a,z);.)$$

taking as given (y,a,r) and (Markov) transition function for $z \to z'$

BELLMAN EQUATION

- **Bellman Equation (for $T \to \infty$)**

$$V(a,z;.) \equiv \max_{c,a'} \left\{ u(c) + \lambda\left(y + (1+r)a - c - a'\right) + \beta \cdot \underset{\substack{\uparrow \\ \text{Expectation in Bellman Equation}}}{E}V(a', \underset{\substack{\uparrow \\ \text{Transition from } z \to z'}}{z'};.) \right\}$$

- **Use to characterize optimal decisions**
- **Current-period FOCs, evaluated using *c(a,z;.)*, *a(a,z;.)***

***c*:**

***a'*:**

Env:

BELLMAN EQUATION

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- **Use to characterize optimal decisions**
- **Current-period FOCs, evaluated using *c(a,z;.)*, *a(a,z;.)***

$$\left.\begin{array}{ll} \textbf{c:} & u'(c(a,z)) - \lambda = 0 \\ \textbf{a':} & -\lambda + \beta E V_1(a(a,z), z(a,z);.) = 0 \\ \textbf{Env:} & EV_1(a,z;.) = \lambda(1+r) \end{array}\right\} \; u'(c(a,z)) = \beta E\left[u'(c(a,z))(1+r)\right]$$

- **Bellman analysis goes through as in deterministic case**
  - **(Given further technical conditions we won't study – Econ 602)**

MARKOV RISK

- Why does Markov assumption make everything work?
- Main issue in moving from deterministic dynamic programming to stochastic dynamic programming: preserving recursivity
  - So exogenous states must also have recursive structure
- Shocks that have this recursive structure are Markov processes
- Markov has property that given the current realization, future realizations are independent of the past
  - "Limited history dependence"
  - "Finite memory"

Macro Fundamentals

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- Shocks that have this recursive structure are Markov processes
- Markov has property that given the current realization, future realizations are independent of the past
  - "Limited history dependence"
  - "Finite memory"
- In environments in which the "regularity conditions" that ensure standard Bellman analysis applies to stochastic problems are not satisfied…
- …often simply need to ASSUME decision rules are Markov to make progress