Department of Economics University of Maryland

Economics 601
Macroeconomic Analysis I
First-Quarter Ph.D. Macro
Problem Set 1 – Suggested Solutions

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Instructions: Written solutions must be submitted no later than 9:30am on the date listed above. Your solutions, which likely require some combination of mathematical derivations, economic reasoning, graphical analysis, and pure logic, should be thoroughly presented and not leave the reader (i.e., the TA and I) guessing about what you actually meant.

You must submit your own independently-written solutions. You are permitted (in fact, encouraged) to work in (small) groups (no larger than three people) to think through issues, ideas, and mechanics; but you must submit your own independently-written solutions, indicating with whom you collaborated. **Under no circumstances will multiple verbatim identical submissions be considered acceptable.**

Solutions should be clearly, logically, and thoroughly presented. Your method of argument(s) and approach to problems is as important as, if not more important than, your "final answer." Throughout, your analysis should be based on the methods and concepts we have developed in class and/or you have studied in related courses.

There are three problems.

Problem 1: Intertemporal Elasticity of Substitution (9 points). Derive the intertemporal elasticity of substitution (IES) for the following utility functions in the deterministic two-period model. Make clear the formal and intuitive definition of the IES. If the IES measure does not exist, explain briefly why.

a. $u(c_1, c_2) = \frac{c_1^{1-\sigma} - 1}{1-\sigma} + \frac{c_2^{1-\sigma} - 1}{1-\sigma}$, $\sigma > 0$.

b. $u(c_1, c_2) = -\frac{1}{a} e^{-ac_1} - \frac{1}{a} e^{-ac_2}$

c. $u(c_1, c_2) = \gamma c_1 - \frac{\alpha}{2} c_1^2 + \gamma c_2 - \frac{\alpha}{2} c_2^2, \gamma > 0, \alpha > 0$

Solution: The IES measure is computed as

$$IES(c_2 / c_1) = \frac{\partial \ln(c_2 / c_1)}{\partial \ln(1 + r_1)}.$$

It measures the intertemporal (i.e., across periods) substitutability of consumption growth in response to change in the (gross) real interest rate, coming from the Euler equation.

a. For this functional form (CRRA in each period), the derivatives are $u_1(c_1, c_2) = c_1^{-\sigma}$ and $u_2(c_1, c_2) = c_2^{-\sigma}$, which can be used to construct the Euler equation: $\frac{c_1^{-\sigma}}{c_2^{-\sigma}} = 1 + r_1$. Computing the natural log (and making sure we handle the exponents correctly), we have $\ln\left(\frac{c_2}{c_1}\right) = \frac{1}{\sigma} \ln(1 + r_1)$. The IES is hence $\frac{1}{\sigma}$.

b. For this functional form (CARA in each period), the derivatives are $u_1(c_1, c_2) = e^{-ac_1}$ and $u_2(c_1, c_2) = e^{-ac_2}$, which can be used to construct the Euler equation: $\frac{e^{-ac_1}}{e^{-ac_2}} = 1 + r_1$, or, taking natural logs of both sides, $a \cdot (c_2 - c_1) = \ln(1 + r_1)$. If we defined $\ln x \equiv c_2 - c_1$, then we could compute $\frac{\partial \ln x}{\partial \ln(1 + r_1)} = \frac{1}{a}$, which looks similar to the CRRA case. What we are looking at, however, is the level of consumption growth – i.e., $\ln x = c_1 \left(\frac{c_2}{c_1} - 1\right)$. So if forced to compute the standard IES for this utility function, this is as far as we can take it.

c. For this functional form (quadratic in each period), the derivatives are $u_1(c_1,c_2)=\gamma-\alpha c_1$ and $u_2(c_1,c_2)=\gamma-\alpha c_2$, which can be used to construct the Euler equation: $\frac{\gamma-\alpha c_1}{\gamma-\alpha c_2}=1+r_1$. Similar to the above, we could simply compute natural logs: $\ln\frac{\gamma-\alpha c_1}{\gamma-\alpha c_2}=\ln\left(1+r_1\right)$, and then define $x\equiv\frac{\gamma-\alpha c_1}{\gamma-\alpha c_2}$. Thus, we have computed the IES = 1. Note a couple of limits we can compute: first, $\lim\limits_{\gamma\to 0}\frac{\partial\ln x}{\partial\ln(1+r_1)}=1$. Second, $\lim\limits_{c_2\to c_1}\frac{\partial\ln x}{\partial\ln(1+r_1)}=1$. What these two limits show is that when the quadratic period utility function has either or both of these properties, the IES converges to the case of 1; when either property is not sufficiently satisfied, then the IES is something much more complicated.

Problem 2: Relative and Absolute Risk Aversion (9 points). Derive measures of relative risk aversion (RRA) and absolute risk aversion (ARA) for the following utility functions in the two-period model (technically, stochastic, but we will soon discuss this further). Make clear the formal and intuitive definition of RRA and ARA. If the RRA and/or ARA measures do not exist, explain briefly why.

a. $u(c_1, c_2) = \frac{c_1^{1-\sigma} - 1}{1-\sigma} + \frac{c_2^{1-\sigma} - 1}{1-\sigma}$, $\sigma > 0$.

b. $u(c_1, c_2) = -\frac{1}{a} e^{-ac_1} - \frac{1}{a} e^{-ac_2}$

c. $u(c_1, c_2) = \gamma c_1 - \frac{\alpha}{2} c_1^2 + \gamma c_2 - \frac{\alpha}{2} c_2^2, \gamma > 0, \alpha > 0$

Solution: In class, we discussed how the RRA(c) and ARA(c) measures are computed as one-period (i.e., static) objects, even though the utility function we are examining are intertemporal. (An important caveat: if we're examining a utility function that is time-non-separable (e.g., habit persistence), then technically the period-t consumption variable shows up in more than one time period's utility function. In this case, it's less easy to say that we are examining the "static" case; let's ignore such cases for now, though it is still the case that the utility function is more like the "static" risk cases.) Note a further distinction between either relative risk aversion measure and the IES measure: the latter is defined over consumption **growth,** while the former is defined over the consumption **level** in a given time period. Thus, in computing the relative risk aversion measures, all that is required is the **level** of consumption (in either period one or period two).

To capture this, we compute the derivatives (first and second) of the $v(.)$ function (with $u(c_1, c_2) = v(c_1) + v(c_2)$). The RRA measure is thus

$$RRA(c) = -\frac{c \cdot v''(c)}{v'(c)}$$

and the ARA measure is

$$ARA(c) = -\frac{v''(c)}{v'(c)}.$$

As discussed in class, the RRA(c) and ARA(c) measures capture the sensitivity of an individual's optimal choice (of consumption) in either relative terms (RRA) or absolute terms (ARA).

a. The derivatives are $v'(c) = c^{-\sigma}$ and $v''(c) = -\sigma c^{-\sigma-1}$. The measure of RRA is thus $RRA(c) = -\frac{c \cdot v''(c)}{v'(c)} = -\frac{-c \cdot \sigma \cdot c^{-\sigma-1}}{c^{-\sigma}} = \sigma$. And the measure of ARA is $ARA(c) = -\frac{v''(c)}{v'(c)} = -\frac{-\sigma c^{-\sigma-1}}{c^{-\sigma}} = \frac{\sigma}{c}$.

b. The derivatives are $v'(c) = e^{-ac}$ and $v''(c) = -ae^{-ac}$. The measure of RRA is thus $RRA(c) = -\frac{c \cdot v''(c)}{v'(c)} = -\frac{-c \cdot ae^{-ac}}{e^{-ac}} = ac$. The measure of ARA is $ARA(c) = -\frac{v''(c)}{v'(c)} = -\frac{-ae^{-ac}}{e^{-ac}} = a$.

c. The derivatives are $v'(c) = \gamma - \alpha c$ and $v''(c) = -\alpha$. The measure of RRA is thus $RRA(c) = -\frac{c \cdot v''(c)}{v'(c)} = -\frac{-c \cdot \alpha}{\gamma - \alpha c} = \frac{c \cdot \alpha}{\gamma - \alpha c}$. The measure of ARA is $ARA(c) = -\frac{v''(c)}{v'(c)} = -\frac{-\alpha}{\gamma - \alpha c} = \frac{\alpha}{\gamma - \alpha c}$.

Problem 3: Arrow-Debreu Securities vs. Non-Arrow-Debreu Securities (15 points). Consider two variations of the stochastic two-period consumption model. Except for what is described below, all other notation and details of the model are exactly as studied in class. In particular, period-2 income has conditional risk characterized by a three-point distribution function *G*(.), with realization y_2^H with probability $1 > q > 0$, realization $\bar{y}_2$ with probability $1 > p > 0$, and realization y_2^L with probability (1-*p*-*q*).

One variation is the model studied in class that has a **single** asset a_1 available for purchase at price *R* in period 1 and that pays a state-contingent return in period 2.

The second variation is that, rather than the single asset a_1, there are **three types of assets** available for purchase. Each unit of asset a_1^H has purchase price R^H in period 1, and pays off one unit of goods in period 2 if state y_2^H is realized and zero in all other realized states; each unit of asset $\bar{a}_1$ has purchase price $\bar{R}$ in period 1, and pays off one unit of goods in period 2 if state $\bar{y}_2$ is realized and zero in all other realized states; and each unit of asset a_1^L has purchase price R^L in period 1, and pays off one unit of goods in period 2 if state y_2^L is realized and zero in all other realized states.

Setting up the budget constraint(s) and Lagrange analysis appropriately, show that the two different asset structures lead to two different allocations.

Solution: In the first case (i.e., the non-AD asset), the Lagrangian for the optimization problem is as studied in class, namely

$$u(c_1) + E_1 u(c_2) + \lambda_1\left[y_1 + (1+r_0)a_0 - c_1 - Ra_1\right] + q\lambda_2^H\left[y_2^H + (1+r_1^H)a_1 - c_2^H\right]$$
$$+p\lambda_2^M\left[\bar{y}_2 + (1+\bar{r}_1)a_1 - c_2^M\right] + (1-p-q)\lambda_2^L\left[y_2^L + (1+r_1^L)a_1 - c_2^L\right]$$

which is the sequential form of the Lagrangian (try it with the lifetime form, too). Note that the $(1 + r_1)$ terms don't matter in the right-hand-sides of the period-2 budget constraints – that is just re-ordering the terms so they are defined in terms of "one-plus…" something (i.e., the *R* price of the single asset is just a definition).

The first-order conditions with respect to c_1, the **triple** of c_2's, and a_1 are thus

$$u'(c_1) - \lambda_1 = 0$$
$$qu'(c_2^H) - q\lambda_2^H = 0$$
$$pu'(c_2^M) - p\lambda_2^M = 0$$
$$(1-p-q)u'(c_2^L) - (1-p-q)\lambda_2^L = 0$$
$$-\lambda_1 R + q\lambda_2^H(1+r_1^H) + p\lambda_2^M(1+\bar{r}_1) + (1-p-q)\lambda_2^L(1+r_1^L) = 0$$

The middle three expressions all say that marginal utility in a particular state in period 2 exactly equals the marginal value of wealth/income. So we can simply focus on the fifth equation. The fifth equation says, rearranged a bit,

$$R = \frac{q\lambda_2^H}{\lambda_1}(1+r_1^H) + \frac{p\lambda_2^M}{\lambda_1}(1+\bar{r}_1) + \frac{(1-p-q)\lambda_2^L}{\lambda_1}(1+r_1^L),$$

or in terms of expectations,

$$R = E_1\left[\frac{\lambda_2}{\lambda_1}(1+r_1)\right].$$

Regardless how it is looked at, this is **one** Euler equation across the two periods.

Turning now to the case of AD assets: the period-1 budget constraint is $c_1 + R^H a_1^H + \bar{R}\,\bar{a}_1 + R^L a_1^L = y_1 + a_0$. The state-contingent period-2 budget constraints are $c_2^j + a_2^j = y_2^j + a_1^j$ for $j \in \{H, L\}$ and $\bar{c}_2 + \bar{a}_2 = \bar{y}_2 + \bar{a}_1$ (in which we can impose the terminal asset holdings for each branch of the event tree, $a_2^H = \bar{a}_2 = a_2^L = 0$).

The Lagrangian for this optimization problem is

$$\begin{aligned} &u(c_1) + E_1 u(c_2) + \lambda_1\left[y_1 + a_0 - c_1 - R^H a_1^H - \bar{R}\,\bar{a}_1 - R^L a_1^L\right] \\ &+q\lambda_2^H\left[y_2^H + a_1^H - c_2^H\right] + p\lambda_2^M\left[\bar{y}_2 + \bar{a}_1 - c_2^M\right] + (1-p-q)\lambda_2^L\left[y_2^L + a_1^L - c_2^L\right] \end{aligned}$$

with Lagrange multipliers for the (three) different possible cases of realized-period-two income, and, importantly, (three) different prices for each of the (three) Arrow-Debreu securities. This makes the optimization problem different.

The first-order conditions with respect to c_1, the **triple** of c_2's, and now the **triple** of a_1's are

$$u'(c_1) - \lambda_1 = 0$$
$$qu'(c_2^H) - q\lambda_2^H = 0$$
$$pu'(c_2^M) - p\lambda_2^M = 0$$
$$(1-p-q)u'(c_2^L) - (1-p-q)\lambda_2^L = 0$$
$$\lambda_1 R^H + q\lambda_2^H = 0$$
$$\lambda_1 \bar{R} + p\lambda_2^M = 0$$
$$\lambda_1 R^L + (1-p-q)\lambda_2^L = 0$$

The last three equations here define **three clearly different** Euler equations: $R^H = \frac{q\lambda_2^H}{\lambda_1}$, $\bar{R} = \frac{p\lambda_2^M}{\lambda_1}$, and $R^L = \frac{(1-p-q)\lambda_2^L}{\lambda_1}$.

In this latter, AD, scenario, each of the three possible realizations of period-2 income can be insured against, so that the individual ends up **equalizing** his marginal utility of period-2 consumption no matter what (presuming that the price of the assets are actuarially fair, which is being assumed here).

Looking instead at the one-asset model, this equalization of marginal utilities cannot in general occur. Equalization of marginal utilities is a basic idea in macro-finance, and AD securities are needed to ensure this result. (And then one can think beyond this to understand how the real world's financial system works – i.e., in many senses, AD assets **are** the financial field's equivalent **over time** of perfect competition in economic markets in a **static sense**.)