Department of Economics University of Maryland

Economics 601
Macroeconomic Analysis I
First-Quarter Ph.D. Macro
Problem Set 2 Suggested Solutions
Professor Sanjay Chugh
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Instructions: Written solutions must be submitted no later than 9:30am on the date listed above. Your solutions, which likely require some combination of mathematical derivations, economic reasoning, graphical analysis, and pure logic, should be thoroughly presented and not leave the reader (i.e., the TA and I) guessing about what you actually meant.

You must submit your own independently-written solutions. You are permitted (in fact, encouraged) to work in (small) groups (no larger than three people) to think through issues, ideas, and mechanics; but you must submit your own independently-written solutions, indicating with whom you collaborated. **Under no circumstances will multiple verbatim identical submissions be considered acceptable.**

Solutions should be clearly, logically, and thoroughly presented. Your method of argument(s) and approach to problems is as important as, if not more important than, your "final answer." Throughout, your analysis should be based on the methods and concepts we have developed in class and/or you have studied in related courses.

There are two problems.

Problem 1: Infinite-Horizon Consumption Model (16 points). Consider the infinite-horizon consumption model starting from the beginning of period zero. Suppose the consumer has lifetime utility function given by

$$\lim_{T\to\infty} E_0 \sum_{t=0}^{T} \beta^t u(c_t)$$

with budget constraints given by $c_0 + \sum_{i=1}^{J} R_{i0} a_0 = y_0 + a_{-1}$ for each t, and state-contingent (i.e., Arrow-Debreu) assets available for purchase in each period, $i \in \{1, 2, ..., J\}$. To make the problem somewhat tractable, suppose that the set J of realizations is large enough that it encompasses all of the risk that could unfold over time. The function $u(.)$ is strictly increasing and strictly concave. (And we are of course considering rational expectations in everything below.)

a. **(2 points)** Construct a recursive problem (i.e., dynamic programming problem) based on the above. State explicitly what other (if any other) assumptions are required in order to make the problem have a unique solution.

Solution: To construct a recursive problem, we require, most importantly, the exogenous processes to be driven by Markov shocks. This is already implied by the structure of the budget constraint written above (which holds for each t): in each period, the state is realized, and there is **only one** asset (called a_{-1} above) that pays off. All of the other assets that were pre-accumulated entering that period are worth zero.

Then, starting in that period, there are R_{it} prices for (a complete set of) new assets (the specific label is R_{i0} above, but this gets updated in every period). We could proceed here with this notation, but to be more careful (and to make things more comparable with the analysis in the sequential setup below), write this asset price as $R_{it}(s_{t+1}|s_t)$. (In which, note, both sets of notation i and s_{t+1} are included, for clarity but also redundancy).

The value function is then given by

$$V(a_{t-1};\cdot) = \max_{c_t,\, a_t(s_{t+1})} \left\{ u(c_t) + \lambda_t \left[y_t + a_{t-1} - c_t - \sum_{s_{t+1}} R_{it}(s_{t+1} \mid s_t) a_t(s_{t+1}) \right] + \beta E_t V(a_t(s_{t+1});\cdot) \right\}.$$

If we want to further emphasize the probabilities of state s_{t+1} given state s_t has occurred, suppose that its probability is $p(s_{t+1}|s_t)$; with this notation, the value function can be written as

$$V(a_{t-1};\cdot) = \max_{c_t,\, a_t(s_{t+1})} \left\{ u(c_t) + \lambda_t \left[y_t + a_{t-1} - c_t - \sum_{s_{t+1}} R_{it}(s_{t+1} \mid s_t) a_t(s_{t+1}) \right] + \beta \sum_{s_{t+1}} V(a_t(s_{t+1});\cdot) p(s_{t+1} \mid s_t) \right\}.$$

In writing this recursive representation, some arbitrary date t is chosen; the date $t = 0$ can specifically be chosen, as well.

b. **(2 points)** Based on the recursive problem above, construct the Euler equation(s) for period t. State any further (if any other) assumptions required to make the Euler equation(s) well-behaved.

Solution: (As above, note that both sets of notation i and s_{t+1} are included, for clarity but also redundancy.) The first-order condition of the recursive problem with respect to (each of the vector of) a_{it} is

$$-\lambda_t R_{it}(s_{t+1} \mid s_t) + \beta V_1(a^*_{it}(s_{t+1});.) p_i(s_{t+1} \mid s_t) = 0$$

for all $i \in \{1, 2, ..., J\}$. The envelope theorem (also known as the **Benveniste-Scheinkman** theorem in dynamic programming) then tells us that $V_1(a^*_{it-1}(s_t);.) = \lambda_{it}(s_t)$ (note here that we are explicitly including the states of period t; simply because updating this one period gives $V_1(a^*_{it}(s_{t+1});.) = \lambda_{it+1}(s_{t+1})$. Putting these conditions together gives $-\lambda_t R_{it}(s_{t+1} \mid s_t) + \beta\lambda_{it+1}(s_{t+1}) p_i(s_{t+1} \mid s_t) = 0$, $i \in \{1, 2, ..., J\}$, or, equivalently,

$$R_{it}(s_{t+1} \mid s_t) = p_{it}(s_{t+1} \mid s_t) \frac{\beta\lambda_{it+1}(s_{t+1})}{\lambda_t}$$

for all $i \in \{1, 2, ..., J\}$. These conditions are the Euler equations, which are expressions evaluated at time t.

Instead, in terms of marginal utility functions for consumption, $R_{it}(s_{t+1} \mid s_t) = p_{it}(s_{t+1} \mid s_t) \frac{\beta u'(c_{it+1}(s_{t+1}))}{u'(c_t)}$ for all $i \in \{1, 2, ..., J\}$. (Either form of the Euler equation was fine.) Besides the assumption of Markov risk (as was already stated in part a), there are no other assumptions required here.

c. **(2 points)** State the complete/proper definition of the equilibrium solution of the recursive problem above. Be sure to include every detail.

Solution: Equilibrium in this recursive version of the problem is a pair of time-invariant functions $\{c(\mathbf{S}_t), a(\mathbf{S}_t)\}$ that jointly satisfy the set of Euler equations above, and the sequence of budget constraints $c_t + \sum_{i=1}^{J} R_{it} a_{it} = y_t + a_{t-1}$. The set $\mathbf{S}_t = \left[y_t, a_{t-1} \right]$ (assuming that it is the set of assets coming into a period and the realization of income that are stochastic). (Also note that we could have specified the value function $V(.)$ as part of the set of objects being determined, with the Bellman equation as the third equations.)

d. **(2 points)** Suppose instead of the recursive problem above, you want to study the sequential problem (i.e., sequential Lagrangian). Construct a sequential problem based on the above. In particular, do **not** make any other assumptions required in order to make the problem have a unique solution.

Solution: The sequential Lagrangian starting from period zero (and in this case, there is no other choice except to start from period zero, simply because the problem is not recursive) is

$$\lim_{T \to \infty} \max_{c_t} \sum_{t=0}^{T} \sum_{s^t} \beta^t \left\{ u(c_t(s^t)) \tilde{p}_t(s^t) + \frac{\lambda_0}{\beta^t} \left[y_t(s^t) - c_t(s^t) \right] q_t(s^t) \right\}.$$

This setup is based on the Arrow-Debreu asset structure described in Ljungqvist and Sargent (Chapter 8.5) (you could have chosen the alternative structure of Arrow assets presented in Chapter 8.8). The term $\tilde{p}_t(s^t)$ (which, note, is distinct from the term p in the recursive structure above), is the price of period- t consumption contingent on history s_t at t, in terms of an abstract unit of account.

To see the terms slightly differently, the utility function is $\lim_{T \to \infty} E_0 \sum_{t=0}^{T} \beta^t u(c_t)$, and the **single** budget constraint is $\lim_{T \to \infty} \sum_{t=0}^{\infty} \sum_{s^t} \left[y_t(s^t) - c_t(s^t) \right] \frac{\lambda_0}{\beta^t} \geq 0$.

Note how much **less** structure the non-Markov problem contains. Nonetheless, in the next part, we will construct state prices (i.e., Euler equations).

e. **(2 points)** Based on the sequential problem above, construct the Euler equation(s) for period t.

Solution: As described in Ljungqvist and Sargent (Chapter 8.5 and 8.7), one can construct one-period-ahead prices based on the chosen optimal decisions. Using their notation, the one-period-ahead Euler equation is

$$\tilde{Q}(s_{t+1} \mid s^t) = \frac{\beta u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \cdot q(s^{t+1} \mid s^t)$$

which requires an assumption about initial-period wealth to make it compatible with the Arrow-Debreu allocation. Given the differences in state-contingent functions in the recursive formulation above, and the truly sequential analysis here, this function looks very similar to the one that appears in part b.

f. **(2 points)** State the complete/proper definition of the equilibrium solution of the sequential problem above. Be sure to include every detail.

Solution: With abuse of notation, in principle, equilibrium in this sequential version of the problem is an infinite-dimensional sequence $\left\{c_t(s^t), a_t(s^{t+1})\right\}_{t=0}^{\infty}$ that jointly satisfy the sets of Euler equations above, and the infinite sequences of budget constraints. Note that each of these objects are vectors of consumption and (if we are allowing it) asset holdings choices.

g. **(2 points)** Construct the value function, carefully labeling variables with their optimum values, of either the recursive problem, the sequential problem, or both.

Solution: For the recursive problem, the value function is as written in part a above, so let's just use that one.

h. **(2 points)** Suppose we are looking to differentiate the value function with respect to parametric assets. Which variable(s), if any variable(s), can we differentiate the value function with respect to? If it exists, differentiate the value function with respect to those variable(s). Provide brief interpretation

Solution: With the structure in place immediately above, it is clear that the envelope condition tells us that $V_1(a_{t-1},\cdot) = \lambda_t$. The immediate interpretation is (the usual interpretation) that when we are starting from an optimal solution, then changing one of the **parameters** of the problem only leads to a "first-order" change in the overall utility (i.e., lifetime utility).

Problem 2: Finite-Horizon Dynamic Programming (17 points). Consider a three-period (period 0, period 1, period 2) **deterministic** consumer problem. The real endowment incomes in periods 0, 1, and 2, are constant at y, and the real interest rate on assets brought into each of the three periods is also constant at r. All of this is known from the beginning of period zero (because the problem is deterministic).

Denote the value functions for periods 0, 1, and 2, respectively, by $V^0(a_{-1})$, $V^1(a_0)$, and $V^2(a_1)$; the state variables that are arguments to the value functions are already provided to you, and there are no other arguments. The value functions for "period 3" and beyond are all zero.

The lifetime utility function of the consumer, starting from the beginning of period zero, is

$$\frac{c_0^{1-\sigma}-1}{1-\sigma}+\frac{\beta(c_1^{1-\sigma}-1)}{1-\sigma}+\frac{\beta^2(c_2^{1-\sigma}-1)}{1-\sigma},$$

in which the scalar $\beta \in (0,1)$ is a standard one-period-ahead subjective discount factor, $\sigma > 0$ is a parameter of the utility function, and, as always, c_t, denotes consumption in periods t = 0, 1, 2. The sequence of budget constraints faced by the consumer are also as usual,

$$\begin{aligned} c_0 + a_0 &= y + (1+r)a_{-1} \\ c_1 + a_1 &= y + (1+r)a_0 \\ c_2 + a_2 &= y + (1+r)a_1 \end{aligned}$$

in which a_t denotes asset holdings at the end of period t. The **terminal condition of this problem is $a_2 = 0$**, and the parameters of the entire lifetime utility maximization problem are (y, r, a_{-1}).

You have the following three (related) tasks:

1. Conduct a **value function iteration to develop expressions for the value functions $V^0(a_{-1})$, $V^1(a_0)$, and $V^2(a_1)$** (where the superscript denotes the period from which $V(.)$ begins).

2. Develop **closed-form expressions for the optimal decision rules (aka policy functions)** for consumption in each of the three periods and (end-of-period) asset holdings for period zero and period one. These policy functions should be functions of only the state variables for any given period. That is, develop closed-form expressions for the optimal choices $c_0^* = c^0(a_{-1})$, $a_0^* = a^0(a_{-1})$, $c_1^* = c^1(a_0)$, $a_1^* = a^1(a_0)$, and $c_2^* = c^2(a_1)$ that depend on only the given state variables and fixed

parameters of the problem. (You do **not** have to obtain closed-form solutions for these optimal choices in terms of only parameters of the problem.)

3. Answer the following: are there any restriction(s) on parameters that make the value functions $V^0(a_{-1}) = V^1(a_0) = V^2(a_1)$? If so, develop the restriction(s) and describe the economic intuition; if not, describe intuitively why there is no such restriction(s)**.** (This question is likely best answered after completing the above.)

Note that while there are several ways one can solve the underlying optimization problem, you are being asked to do so via a value function iteration (i.e., you are asked to demonstrate conducting a value function iteration).

Solution: In a few steps, we'll consider just the case of $\sigma = 1$ (the case of log utility). Not because the $\sigma \neq 1$ case is irrelevant, but simply because the $\sigma = 1$ case can be solved in closed form. So your solutions should be checked with this limiting case.

The problem requires backward iteration. The (trivial) Bellman equation starting from period 2 is

$$V^2(a_1) = \max_{c_2} \left\{ u(c_2) + \lambda_2 \left[y + (1+r)a_1 - c_2 \right] \right\},$$

in which the terminal condition $a_2 = 0$ is imposed and there is no continuation value function (because there is no period three). Using the functional form for period utility, the straightforward optimization leads to $c_2^* = y + (1+r)a_1 \equiv c^2(a_1)$, which is the consumption policy function for period 2.

Inserting this into the period-2 Bellman equation **and** recognizing that the period-2 budget constraint holds with equality at the optimal choice, the period-2 value function is

$$V^2(a_1) = \frac{\left(c^2(a_1)\right)^{1-\sigma} - 1}{1-\sigma} + \lambda^2(a_1)\left[y + (1+r)a_1 - c^2(a_1) \right],$$

which has associated envelope condition

$$V_a^2(a_1) = \lambda^2(a_1)(1+r) = u'(c^2(a_1))(1+r) = \frac{1+r}{\left[c^2(a_1)\right]^{\sigma}} = \frac{1+r}{\left[y + (1+r)a_1\right]^{\sigma}}.$$

Iterating backwards, the Bellman equation starting from period 1 is

$$V^1(a_0) = \max_{c_1, a_1} \left\{ u(c_1) + \lambda_1 \left[y + (1+r)a_0 - c_1 - a_1 \right] + \beta V^2(a_1) \right\}.$$

The first-order conditions with respect to c_1 and a_1 give $\frac{1}{c_1^{\sigma}} = \lambda_1$ and $\lambda_1 = \beta V_a^2(a_1)$.

From here on, we limit attention to the case of $\sigma = 1$ because that allows closed-form solutions; in general, though, one could continue with the $\sigma \neq 1$ case.

Substituting the period-2 envelope condition (now with the limiting case of $\sigma = 1$ considered), we have the period-1 Euler equation

$$\frac{1}{c_1^*} = \frac{\beta(1+r)}{y+(1+r)a_1^*},$$

with asterisks now written to emphasize we are looking at the optimal solution. The other condition that relates c_1^* and a_1^* is the period-1 budget constraint, $c_1^* = y + (1+r)a_0 - a_1^*$. Solving this simultaneously with the period-1 Euler equation gives

$$c_1^* = \left(\frac{\beta}{1+\beta}\right)\left[\frac{(2+r)y}{\beta(1+r)} + \frac{(1+r)a_0}{\beta}\right] \equiv c^1(a_0)$$

and

$$a_1^* = \left[\frac{(1+\beta)(1+r)-(2+r)}{(1+\beta)(1+r)}\right]y + \frac{\beta}{1+\beta}(1+r)a_0 \equiv a^1(a_0),$$

which are the policy functions for consumption and asset holdings in period 1.

Inserting these period-1 policy functions into the period-1 Bellman equation, the period-1 value function is

$$V^1(a_0) = \frac{\left[c^1(a_0)\right]^{1-\sigma} - 1}{1-\sigma} + \lambda^1(a_0)\left[y + (1+r)a_0 - c^1(a_0) - a^1(a_0)\right] + \beta V^2\left(a^1(a_0)\right),$$

which has associated envelope condition

$$V_a^1(a_0) = \lambda^1(a_0)(1+r) = u'(c^1(a_0))(1+r) = \frac{1+r}{\left[c^1(a_0)\right]^{\sigma}} = \frac{1+r}{\left[y + \left(\frac{1+r}{2+r}\right)(1+r)a_0\right]^{\sigma}}$$

(in which we will, again, consider the limit case of $\sigma = 1$). Iterating backwards again, the Bellman equation starting from period 0 is

$$V^0(a_{-1}) = \max_{c_0, a_0}\left\{u(c_0) + \lambda_0\left[y + (1+r)a_{-1} - c_0 - a_0\right] + \beta V^1(a_0)\right\}.$$

The first-order conditions with respect to c_0 and a_0 give $\frac{1}{c_0^{\sigma}} = \lambda_0$ and $\lambda_0 = \beta V_a^1(a_0)$. Substituting the period-1 envelope condition (evaluated at σ = 1), we have the period-0 Euler equation

$$\frac{1}{c_0^*} = \frac{\beta(1+r)}{y + \left(\frac{1+r}{2+r}\right)(1+r)a_0^*},$$

with asterisks now written to emphasize we are looking at the optimal solution. The other condition that relates c_0^* and a_0^* is the period-0 budget constraint, $c_0^* = y + (1+r)a_{-1} - a_0^*$. Solving this simultaneously with the period-0 Euler equation gives, after some algebra,

$$c_0^* = \left(\frac{1+(1+r)}{\beta + \beta(1+r) + (1+r)}\right)y + \left(\frac{(1+r)(1+r)}{\beta + \beta(1+r) + (1+r)}\right)a_{-1} \equiv c^0(a_{-1})$$

and

$$a_0^* = \left(1 - \frac{1+(1+r)}{\beta + \beta(1+r) + (1+r)}\right)y + \left(1 + r - \frac{(1+r)(1+r)}{\beta + \beta(1+r) + (1+r)}\right)a_{-1} \equiv a^0(a_{-1}),$$

which are the policy functions for consumption and asset holdings in period 0. (You may not have grouped terms in exactly this way, which is fine.)

Inserting these period-0 policy functions into the period-0 Bellman equation, the period-0 value function is

$$V^0(a_{-1}) = \frac{\left[c^0(a_{-1})\right]^{1-\sigma} - 1}{1-\sigma} + \lambda^0(a_{-1})\left[y + (1+r)a_{-1} - c^0(a_{-1}) - a^0(a_{-1})\right] + \beta V^1\left(a^0(a_{-1})\right)$$

which we can restate as (because the flow budget constraint holds with equality in each period)

$$V^0(a_{-1}) = \frac{\left[c^0(a_{-1})\right]^{1-\sigma} - 1}{1-\sigma} + \beta V^1\left(a^0(a_{-1})\right).$$

Indeed, in the final expressions for each of the value functions, we can drop the term associated with the flow budget constraint because the budget constraint holds with equality.

Clearly, there is no way for the value functions (and hence the policy functions) to be time-invariant (which is the next question you are asked) because this is a finite-horizon programming problem. **However,** that does not mean that the consumption **choices** cannot be time-invariant. Indeed, **if we had** $\beta(1+r)=1$, then intertemporal consumption smoothing is achieved here, in the sense that

$$c^0(a_{-1}) = c^1(a^0(a_{-1})) = c^2(a^1(a^0(a_{-1}))) .$$

In a deterministic finite-horizon problem (and given $\beta(1+r)=1$), what supports intertemporal consumption smoothing is a equi-proportional drawdown of assets over time (or, if $a_{-1}<0$, a equi-proportional payback of debt over time), so that at the end of the planning horizon, the terminal condition $a_2 = 0$ is achieved.

Without $\beta(1+r)=1$, the intertemporal consumption path is either upward sloping or downward sloping.