

LINEAR APPROXIMATION OF THE BASELINE RBC MODEL

SEPTEMBER 13, 2011

Introduction

LINEARIZATION OF THE RBC MODEL

- For $f(x, y, z) = 0$, multivariable Taylor linear expansion around $(\bar{x}, \bar{y}, \bar{z})$

$$f(x, y, z) \approx f(\bar{x}, \bar{y}, \bar{z}) + f_x(\bar{x}, \bar{y}, \bar{z})(x - \bar{x}) + f_y(\bar{x}, \bar{y}, \bar{z})(y - \bar{y}) + f_z(\bar{x}, \bar{y}, \bar{z})(z - \bar{z})$$

- Four equations describe the dynamic solution to RBC model

- Consumption-leisure efficiency condition

$$-\frac{u_n(c_t, n_t)}{u_c(c_t, n_t)} = z_t m_n(k_t, n_t)$$

- Consumption-investment efficiency condition

$$u_c(c_t, n_t) = \beta E_t [u_c(c_{t+1}, n_{t+1})(1 - \delta + z_{t+1} m_k(k_{t+1}, n_{t+1}))]$$

- Aggregate resource constraint

$$c_t + k_{t+1} - (1 - \delta)k_t = z_t m(k_t, n_t)$$

- Law of motion for TFP

$$\ln z_{t+1} = (1 - \rho_z) \ln \bar{z} + \rho_z \ln z_t + \varepsilon_{t+1}^z$$

STEADY STATE

- Deterministic steady state the natural local point of approximation
- Shut down all shocks and set exogenous variables at their means
- **The Idea:** Let economy run for many (infinite) periods
 - Time eventually “doesn’t matter” any more
 - Drop all time indices

$$-\frac{u_n(\bar{c}, \bar{n})}{u_c(\bar{c}, \bar{n})} = \bar{z}m_n(\bar{k}, \bar{n})$$

$$u_c(\bar{c}, \bar{n}) = \beta u_c(\bar{c}, \bar{n}) \left[m_k(\bar{k}, \bar{n}) + 1 - \delta \right]$$

$$\bar{c} + \delta \bar{k} = \bar{z}m(\bar{k}, \bar{n})$$

$$\ln \bar{z} = (1 - \rho_z) \ln \bar{z} + \rho_z \ln \bar{z} \Rightarrow \bar{z} = \bar{z} \quad (\text{a parameter of the model})$$

- Given functional forms and parameter values, solve for (c, n, k)
 - **The steady state of the model**
 - **Taylor expansion around this point**

September 13, 2011

3

LINEARIZATION ALGORITHMS

- Schmitt-Grohe and Uribe (2004 *JEDC*)
 - A **perturbation** algorithm
 - A class of methods used to find an **approximate** solution to a problem that cannot be solved exactly, **by starting from the exact solution of a related problem**
 - Applicable if the problem can be formulated by adding a “small” term to the description of the exactly-solvable problem
 - Matlab code available through Columbia Dept. of Economics web site
- Uhlig (1999, chapter in *Computational Methods for the Study of Dynamic Economies*)
 - Uses a generalized eigen-decomposition
 - Typically implemented with Schur decomposition (Sims algorithm)
 - Matlab code available at
<http://www2.wiwi.hu-berlin.de/institute/wpol/html/toolkit.htm>

September 13, 2011

4

LINEARIZATION OF THE RBC MODEL

Define **co-state** vector and **state** vector

$$y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix} \quad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$

LINEARIZATION OF THE RBC MODEL

Define **co-state** vector and **state** vector

$$y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix} \quad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$

Order model's dynamic equations in a **vector** $\equiv f(y_{t+1}, y_t, x_{t+1}, x_t) = 0$

Consumption-leisure efficiency condition

Consumption-investment efficiency condition

Aggregate resource constraint

Law of motion for TFP

$$\begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{bmatrix}$$

LINEARIZATION OF THE RBC MODEL

Need four **matrices** of derivatives

1. Differentiate $f(y_{t+1}, y_t, x_{t+1}, x_t)$ with respect to (elements of) y_{t+1}

	First derivatives with respect to:		
	c_{t+1}	n_{t+1}	
Consumption-leisure efficiency condition			= $f_{y_{t+1}}$
Consumption-investment efficiency condition			
Aggregate resource constraint			
Law of motion for TFP			

September 13, 2011

7

LINEARIZATION OF THE RBC MODEL

Need four **matrices** of derivatives

2. Differentiate $f(y_{t+1}, y_t, x_{t+1}, x_t)$ with respect to (elements of) y_t

	First derivatives with respect to:		
	c_t	n_t	
Consumption-leisure efficiency condition			= f_{y_t}
Consumption-investment efficiency condition			
Aggregate resource constraint			
Law of motion for TFP			

September 13, 2011

8

LINEARIZATION OF THE RBC MODEL

Need four **matrices** of derivatives

3. Differentiate $f(y_{t+1}, y_t, x_{t+1}, x_t)$ with respect to (elements of) x_{t+1}

First derivatives with respect to:

	k_{t+1}	z_{t+1}	
Consumption-leisure efficiency condition	$\begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$		$= f_{x_{t+1}}$
Consumption-investment efficiency condition			
Aggregate resource constraint			
Law of motion for TFP			

September 13, 2011

9

LINEARIZATION OF THE RBC MODEL

Need four **matrices** of derivatives

4. Differentiate $f(y_{t+1}, y_t, x_{t+1}, x_t)$ with respect to (elements of) x_t

First derivatives with respect to:

	k_t	z_t	
Consumption-leisure efficiency condition	$\begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$		$= f_{x_t}$
Consumption-investment efficiency condition			
Aggregate resource constraint			
Law of motion for TFP			

September 13, 2011

10

LINEARIZATION OF THE RBC MODEL

The model's dynamic **expectational** equations

$$E_t \begin{bmatrix} f^1(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^2(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^3(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^4(y_{t+1}, y_t, x_{t+1}, x_t) \end{bmatrix} = E_t \begin{bmatrix} \text{Consumption-leisure efficiency condition} \\ \text{Consumption-investment efficiency condition} \\ \text{Aggregate resource constraint} \\ \text{Law of motion for TFP} \end{bmatrix}$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic **expectational** equations

$$E_t \begin{bmatrix} f^1(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^2(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^3(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^4(y_{t+1}, y_t, x_{t+1}, x_t) \end{bmatrix} = E_t \begin{bmatrix} \text{Consumption-leisure efficiency condition} \\ \text{Consumption-investment efficiency condition} \\ \text{Aggregate resource constraint} \\ \text{Law of motion for TFP} \end{bmatrix}$$

Conjecture equilibrium decision rules

Note: $g(\cdot)$ and $h(\cdot)$ are time invariant functions!

$$y_t = g(x_t, \sigma)$$

$$x_{t+1} = h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}$$

Substitute decision rules
into dynamic equations

“Perturbation parameter”:
governs size of shocks

Matrix of standard
deviations of state
variables

LINEARIZATION OF THE RBC MODEL

The model's dynamic **expectational** equations

$$\begin{aligned}
 E_t [f(y_{t+1}, y_t, x_{t+1}, x_t)] &= 0 \\
 &= E_t [f(g(x_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}, x_t)] \\
 &= E_t [f(g(h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}, x_t)] \\
 &\equiv F(x_t, \sigma)
 \end{aligned}$$



$$F_x(x_t, \sigma) = 0 \quad F_\sigma(x_t, \sigma) = 0$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic **expectational** equations

$$\begin{aligned}
 E_t [f(y_{t+1}, y_t, x_{t+1}, x_t)] &= 0 \\
 &= E_t [f(g(x_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}, x_t)] \\
 &= E_t [f(g(h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}, x_t)] \\
 &\equiv F(x_t, \sigma)
 \end{aligned}$$



Using chain rule and
suppressing arguments

$$F_x(x_t, \sigma) =$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic **expectational** equations

$$\begin{aligned}
 E_t [f(y_{t+1}, y_t, x_{t+1}, x_t)] &= 0 \\
 &= E_t [f(g(x_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}, x_t)] \\
 &= E_t [f(g(h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}, x_t)] \\
 &\equiv F(x_t, \sigma)
 \end{aligned}$$



Using chain rule and
suppressing arguments

$$\begin{aligned}
 F_x(x_t, \sigma) &= f_{y_{t+1}} \cdot g_x \cdot h_x + f_{y_t} \cdot g_x + f_{x_{t+1}} \cdot h_x + f_{x_t} \\
 &= 0
 \end{aligned}$$

September 13, 2011

15

LINEARIZATION OF THE RBC MODEL

The model's dynamic **expectational** equations

$$\begin{aligned}
 E_t [f(y_{t+1}, y_t, x_{t+1}, x_t)] &= 0 \\
 &= E_t [f(g(x_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}, x_t)] \\
 &= E_t [f(g(h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}, x_t)] \\
 &\equiv F(x_t, \sigma)
 \end{aligned}$$



Using chain rule and
suppressing arguments

$$\begin{aligned}
 F_x(x_t, \sigma) &= f_{y_{t+1}} \cdot g_x \cdot h_x + f_{y_t} \cdot g_x + f_{x_{t+1}} \cdot h_x + f_{x_t} \\
 &= 0
 \end{aligned}$$

Holds, in particular, at the deterministic steady state $(\bar{x}, 0)$

$$F_x(\bar{x}, 0) = f_{y_{t+1}} \cdot g_x \cdot h_x + f_{y_t} \cdot g_x + f_{x_{t+1}} \cdot h_x + f_x = 0$$

Setting $\sigma = 0$ shuts
down shocks

Each term is evaluated at
the steady state – just as
Taylor theorem requires

September 13, 2011

16

LINEARIZATION OF THE RBC MODEL

- A **quadratic** equation in the elements of g_x and h_x evaluated at the steady state
- $$F_x(\bar{x}, 0) = f_{y_{t+1}}(\bar{x}, 0) \cdot g_x(\bar{x}, 0) \cdot h_x(\bar{x}, 0) + f_{y_t}(\bar{x}, 0) \cdot g_x(\bar{x}, 0) + f_{x_{t+1}}(\bar{x}, 0) \cdot h_x(\bar{x}, 0) + f_{x_t}(\bar{x}, 0) = 0$$
- Solve numerically for the elements of g_x and h_x (use **fsolve** in Matlab)
 - Recall conjectured equilibrium decision rules

$$y_t = g(x_t, \sigma)$$

$$x_{t+1} = h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}$$

September 13, 2011

17

LINEARIZATION OF THE RBC MODEL

- A **quadratic** equation in the elements of g_x and h_x evaluated at the steady state
- $$F_x(\bar{x}, 0) = f_{y_{t+1}}(\bar{x}, 0) \cdot g_x(\bar{x}, 0) \cdot h_x(\bar{x}, 0) + f_{y_t}(\bar{x}, 0) \cdot g_x(\bar{x}, 0) + f_{x_{t+1}}(\bar{x}, 0) \cdot h_x(\bar{x}, 0) + f_{x_t}(\bar{x}, 0) = 0$$
- Solve numerically for the elements of g_x and h_x (use **fsolve** in Matlab)
 - Recall conjectured equilibrium decision rules

$$y_t = g(x_t, \sigma)$$

$$x_{t+1} = h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}$$

- First-order approximation is

$$y_t = g(x_t, \sigma) \approx g(\bar{x}, 0) + g_x(\bar{x}, 0)(x_t - \bar{x}) + \overset{=0}{g_\sigma(\bar{x}, 0)\sigma} \quad \text{SGU Theorem 1:}$$

$$x_{t+1} = h(x_t, \sigma) \approx h(\bar{x}, 0) + h_x(\bar{x}, 0)(x_t - \bar{x}) + \overset{=0}{h_\sigma(\bar{x}, 0)\sigma} \quad g_\sigma = 0 \text{ and } h_\sigma = 0$$

- **DONE!!!**

- Now conduct impulse responses, tabulate business cycle moments, write paper

September 13, 2011

18

CERTAINTY EQUIVALENCE

- Displayed by a model if decision rules do **not** depend on the standard deviation of exogenous uncertainty – e.g., **PRECAUTIONARY SAVINGS!**

- For **stochastic** problems with **quadratic objective function** and **linear constraints**, the decision rules are identical to those of the nonstochastic problem

- Here, we have

$$y_t = g(x_t, \sigma) \approx g(\bar{x}, 0) + g_x(\bar{x}, 0)(x_t - \bar{x}) + \overset{=0}{g_\sigma(\bar{x}, 0)\sigma}$$

$$x_{t+1} = h(x_t, \sigma) \approx h(\bar{x}, 0) + h_x(\bar{x}, 0)(x_t - \bar{x}) + \overset{=0}{h_\sigma(\bar{x}, 0)\sigma}$$

- SGU Theorem 1:** $g_\sigma = 0$ and $h_\sigma = 0$
 - First-order approximated decision rules do not depend on the size of the shocks, which is governed by σ
 - Not the same thing as “**exact CE**,” but refer to it as CE

September 13, 2011

19

LINEARIZING THE RBC MODEL

- Assume $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ and $m(k_t, n_t) = k_t^\alpha n_t^{1-\alpha}$
- $\therefore$ consumption-leisure efficiency condition is $\frac{\psi c_t}{n_t} - (1-\alpha)z_t k_t^\alpha n_t^{-\alpha} = 0$
- Let $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \frac{\psi c_t}{n_t} - (1-\alpha)z_t k_t^\alpha n_t^{-\alpha} = 0$ (and recall $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$)

September 13, 2011

20

LINEARIZING THE RBC MODEL

- Assume $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ and $m(k_t, n_t) = k_t^\alpha n_t^{1-\alpha}$
- $\therefore$ consumption-leisure efficiency condition is $\frac{\psi/c_t}{n_t} - (1-\alpha)z_t k_t^\alpha n_t^{-\alpha} = 0$
- Let $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \frac{\psi/c_t}{n_t} - (1-\alpha)z_t k_t^\alpha n_t^{-\alpha} = 0$ (and recall $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$)
- Compute first row of matrix $f_{y,t+1}$

	c_{t+1}	n_{t+1}
Consumption-leisure efficiency condition	$\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$	
Consumption-investment efficiency condition		
Aggregate resource constraint		
Law of motion for TFP		

September 13, 2011

21

LINEARIZING THE RBC MODEL

- Assume $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ and $m(k_t, n_t) = k_t^\alpha n_t^{1-\alpha}$
- $\therefore$ consumption-leisure efficiency condition is $\frac{\psi/c_t}{n_t} - (1-\alpha)z_t k_t^\alpha n_t^{-\alpha} = 0$
- Let $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \frac{\psi/c_t}{n_t} - (1-\alpha)z_t k_t^\alpha n_t^{-\alpha} = 0$ (and recall $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$)
- Compute first row of matrix $f_{y,t+1}$

	c_{t+1}	n_{t+1}
Consumption-leisure efficiency condition	$\begin{bmatrix} 0 & 0 \\ & & & \\ & & & \\ & & & \end{bmatrix}$	
Consumption-investment efficiency condition		
Aggregate resource constraint		
Law of motion for TFP		

September 13, 2011

22

LINEARIZING THE RBC MODEL

- Assume $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ and $m(k_t, n_t) = k_t^\alpha n_t^{1-\alpha}$
- $\therefore$ consumption-leisure efficiency condition is $\frac{\psi c_t}{n_t} - (1-\alpha)z_t k_t^\alpha n_t^{-\alpha} = 0$
- Let $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \frac{\psi c_t}{n_t} - (1-\alpha)z_t k_t^\alpha n_t^{-\alpha} = 0$ (and recall $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$)
- Compute first row of matrix f_{yt}

	c_t	n_t
Consumption-leisure efficiency condition	$\frac{\psi}{n_t}$	$-\frac{\psi c_t}{n_t^2} + \alpha(1-\alpha)z_t k_t^\alpha n_t^{-\alpha-1}$
Consumption-investment efficiency condition		
Aggregate resource constraint		
Law of motion for TFP		

September 13, 2011

23

LINEARIZING THE RBC MODEL

- Assume $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ and $m(k_t, n_t) = k_t^\alpha n_t^{1-\alpha}$
- $\therefore$ consumption-leisure efficiency condition is $\frac{\psi c_t}{n_t} - (1-\alpha)z_t k_t^\alpha n_t^{-\alpha} = 0$
- Let $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \frac{\psi c_t}{n_t} - (1-\alpha)z_t k_t^\alpha n_t^{-\alpha} = 0$ (and recall $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$)
- Compute first row of matrix f_{xt+1}

	k_{t+1}	z_{t+1}
Consumption-leisure efficiency condition	0	0
Consumption-investment efficiency condition		
Aggregate resource constraint		
Law of motion for TFP		

September 13, 2011

24

LINEARIZING THE RBC MODEL

- Assume $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ and $m(k_t, n_t) = k_t^\alpha n_t^{1-\alpha}$
- $\therefore$ consumption-leisure efficiency condition is $\frac{\psi c_t}{n_t} - (1-\alpha)z_t k_t^\alpha n_t^{-\alpha} = 0$
- Let $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \frac{\psi c_t}{n_t} - (1-\alpha)z_t k_t^\alpha n_t^{-\alpha} = 0$ (and recall $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$)
- Compute first row of matrix f_{xt}

	k_t	z_t
Consumption-leisure efficiency condition	$-\alpha(1-\alpha)z_t \frac{k_t^{\alpha-1}}{n_t^\alpha}$	$-(1-\alpha) \frac{k_t^\alpha}{n_t^{\alpha+1}}$
Consumption-investment efficiency condition		
Aggregate resource constraint		
Law of motion for TFP		

September 13, 2011

25

LINEARIZING THE RBC MODEL

- In deterministic steady state, the first rows of $f_{y,t+1}$, $f_{y,t}$, $f_{x,t+1}$, $f_{x,t}$ are

	k	z
$f_{y,t+1}$	0	0
$f_{y,t}$	$\frac{\psi}{\bar{n}}$	$-\frac{\psi \bar{c}}{\bar{n}^2} + \alpha(1-\alpha)\bar{z}\bar{k}^\alpha \bar{n}^{-\alpha-1}$
$f_{x,t+1}$	0	0
$f_{x,t}$	$-\alpha(1-\alpha)\bar{z} \frac{\bar{k}^{\alpha-1}}{\bar{n}^\alpha}$	$-(1-\alpha) \frac{\bar{k}^\alpha}{\bar{n}^{\alpha+1}}$
- How to compute derivatives $f_{y,t+1}$, $f_{y,t}$, $f_{x,t+1}$, $f_{x,t}$?
 - By hand (feasible for small models)
 - Schmitt-Grohe and Uribe Matlab analytical routines
 - Your own Maple or Mathematica programs
 - Dynare package

September 13, 2011

26

CALIBRATION?

- Solving for the steady state?
- Choosing parameter values?
- Next: calibration of the baseline RBC model