Chapter 3
Dynamic Consumption-Savings Framework

We just studied the consumption-leisure model as a "one-shot" model in which individuals had no regard for the future: they simply worked to earn income, all of which they then spent on consumption right away, putting away none of it for the future.

Individuals do, of course, consider their future prospects when making economic decisions about the present. When an individual makes his or her optimal choices about consumption and leisure in the current period, he/she usually recognizes that he/she will make a similar consumption-leisure choice in the future. In effect, then, it seems there are multiple consumption-leisure choices an individual makes over the course of his/her lifetime.

However, these choices are not independent of each other because **consumers can save for the future or borrow against future income** (borrowing is simply negative savings, also known as **"dissaving"**). That is, current choices affect future choices, and, conversely, (expectations of) future events and choices affect current choices.

In this section, we will focus on the study of **intertemporal** (literally, "across time") choices of individuals. The easiest way to understand the basics of intertemporal choice theory is by first ignoring leisure and labor altogether. That is, we will revert to our assumption that an individual has no control over his or her income.

Rather, we will enrich our model of consumer theory by now supposing that each individual plans economic events for two time periods – the "present" period and the "future" period. We will designate the present period as "period 1" and the future period as "period 2." There is no "period 3" in the economic planning horizon, and every individual knows there is no period 3.[25] This stark division of all time into just two periods will serve to illustrate the basic principles of (macro)economic events unfolding as a sequence over time; after mastering the basics of ***dynamic macroeconomics*** by using the two-period model, we will eventually extend to consideration of an infinite-period model, which arguably may be more realistic because, after all, when does time "end?" But let's build that up slowly.

In the two-period model, our stylized (that is, representative) individual will receive "labor income" (over which he/she has no control) in each of the two periods, and has to make a choice about consumption in each of the two periods. Savings or borrowings are allowed during period 1. The notation we will use here, indeed the entire method of analysis, should remind you of our initial study of consumer theory.

[25] Think of this as meaning that the world (and hence the economy) ends with certainty after two periods.

A Simple Intertemporal Utility Function

As always, in order to study consumer choice, we need to first specify the individual's utility function. In our present intertemporal context, the two arguments to the utility function are consumption in period 1 and consumption in period 2, which we will denote by c_1 and c_2, respectively.[26] We will assume all the usual properties of utility functions: utility is always strictly increasing in both arguments and always displays diminishing marginal utility in both arguments. In abstract form, we (again!) will write this utility function as $u(c_1, c_2)$, and the utility function can be represented by an indifference map featuring downward-sloping indifference curves that are bowed in towards the origin.

In everything that follows, we will continue to write $u(c_1, c_2)$ to stand for the intertemporal, or lifetime, utility function. To dip our feet a bit into macroeconomics, though, a commonly-used intertemporal utility function is

$$u(c_1, c_2) = \ln c_1 + \ln c_2,$$

in which "ln" stands for the natural logarithm. The indifferences curves are plotted in three-dimensional space in Figure 18 and in two-dimensional coordinates in Figure 19. Both Figure 18 and Figure 19 should remind you of basic micro concepts.

[26] With this choice of notation, you can already start to see the parallels between the intertemporal consumption model and our initial study of consumer theory. Keep in mind the different interpretation here though, that of intertemporal choice.

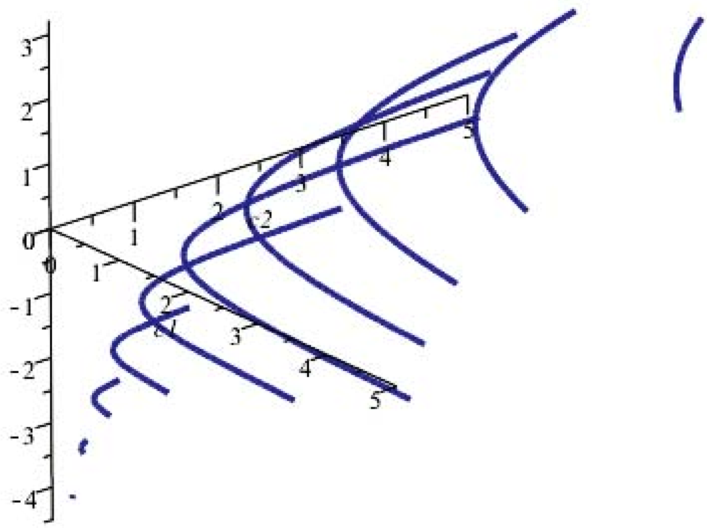

Figure 18. An indifference map of the utility function $u(c_1, c_2) = \ln(c_1) + \ln(c_2)$, where each solid curve represents a given (positive or negative) height above the c_1-c_2 plane and hence a particular level of utility. The three axes are the c_1 axis, the c_2 axis, and the utility axis.

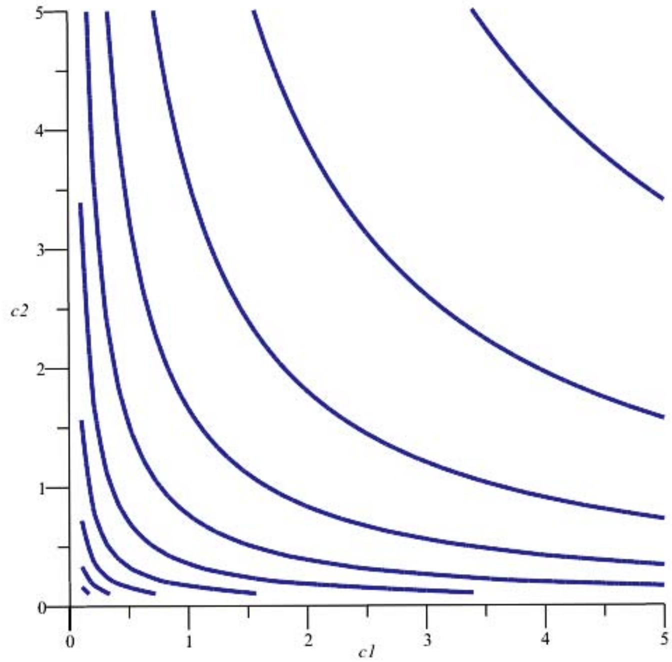

Figure 19. The contours of the utility function $u(c_1, c_2) = \ln(c_1) + \ln(c_2)$ viewed in the two-dimensional c_1-c_2 plane. The utility axis is coming perpendicularly out of the page at you. Each contour is called an indifference curve. Indifference curves further to the northeast are associated with higher levels of utility.

Budget Constraints

The most important way in which the intertemporal consumption model differs from our model of consumer theory heretofore is in the budget constraint(s). Before describing the model further, we need to distinguish between income and wealth, two conceptually different economic ideas.

Income Vs. Wealth

Income is a receipt of money by an individual during some period of time – the most common forms of income are **labor income** (money earned by working) and **interest income** (money earned on assets). On the other hand, an individual's wealth is the level of assets (cash, checking accounts, savings accounts, stock, bonds, etc.) an individual has in store. An individual's wealth may be negative, for example if he is overdrawn on his checking account or otherwise is in debt.

A simple example will illustrate the point. If you currently have $1,000 in your savings account, an economist would say that you have $1,000 in wealth. Say your savings account pays three percent interest per year. If you leave your funds in your savings account alone for the next one year (making neither deposits nor withdrawals), at the end of one year you will have $(1+0.03)\cdot \$1,000 = \$1,030$ in your account. This amount can be decomposed into $1,000 of wealth and $30 of interest income**.** Suppose during that year you also earned $10,000 by working – this amount, not surprisingly, we would call your labor income. Thus, your total income during the year is the sum of your labor income and interest income, in this case $10,030. The $1,000 still in your savings account is **not** part of your income, although it was the basis of your $30 of interest income.

Period-by-Period Budget Constraints

Returning to the description of the two-period model: individuals receive labor income twice in their lives – once in period one and again in period two. As we said above, for now, the amounts of labor income are outside the control of the individual. Soon, we will relax this assumption and allow the individual to have some control over how much labor income he earns. In describing the sequence of economic events, we will need to introduce several elements of notation. The individual receives labor income Y_1 dollars at the beginning of period 1. In addition, the individual begins period 1 with some initial wealth (which may be negative), which we denote by A_0 — we make no assertion about where this initial wealth came from (perhaps it was bequeathed to him by his ancestors). Regardless of where this initial wealth (or initial debt if A_0 is negative) came from, in period 1 it becomes available to the individual along with some nominal interest income.

He chooses consumption c_1 in period 1, each unit of which costs P_1 dollars. He also decides how much wealth to carry into period 2. Denote this level of wealth A_1.

To emphasize, A_1 is chosen in period 1 and is the amount of dollars the individual carries with him (in a savings account, say) from period 1 into period 2. **Notice that A_1 may be negative,** just as A_0 may be negative. A negative A_1 means that the individual is in debt at the beginning of period 2. With this notation, we can write down the **period-1 budget constraint** of the individual

$$P_1 c_1 + A_1 = (1+i)A_0 + Y_1, \tag{6}$$

where i denotes the nominal interest rate (we will say more about this shortly). An **equivalent** version of the period-1 budget constraint is obtained by subtracting P_1c_1 from both sides, which gives

$$A_1 = (1+i)A_0 + Y_1 - P_1 c_1$$

This equivalent expression of the period-1 budget constraint emphasizes that out of all the resources that were available for the first period, A_1, that were **NOT** spent on period-1 consumption and thus carries over to the next period.

At the beginning of period 2, the individual receives nominal income Y_2. If he chose to carry positive wealth A_1 from period 1 into period 2, he receives back (from his bank account, say) the full amount A_1 plus interest earned on that amount. Denote this **nominal interest rate** by i, where $0 \le i \le 1$. For our purposes, t**he nominal interest rate is the return on each dollar kept in a bank account from one period to the next.**

We need to be very clear about the events occurring here, so to re-emphasize: if the individual chose to carry a positive amount A_1 dollars from period 1 into period 2, he receives at the beginning of period 2 his original A_1 dollars plus another iA_1 dollars in interest. On the other hand, if the individual chose to carry a negative A_1 into period 2 (that is, the individual is in debt at the beginning of period 2), he must repay A_1 (to, say, the bank to whom he is in debt) with an interest rate of i – that is, he would repay $A_1 + iA_1$.[27] This nominal interest rate i is the same interest rate that appears in the period-1 budget constraint in expression (6).

[27] For simplicity we are supposing that the interest rate at which the individual can save is the same as the interest rate at which the individual can borrow. In general, this need not and usually is not the case. More generally, we can say that there is an interest rate i_s which the individual would receive if he had a positive level of wealth and a different interest rate i_b which the individual would face if he had a negative level of wealth.

After settling his accounts, the individual then chooses consumption c_2 in period 2, each unit of which costs P_2 dollars. He also decides how much wealth to carry into period 3. Denote this level of wealth by A_2. But the economy ends at the end of period 2 and every individual knows the economy ends at the end of period 2! Thus, there is no period 3 to save for, and no rational bank would allow anyone to die in debt to it – so we must have that $A_2 = 0$.

With this notation, we can write down the **period-2 budget constraint** of the individual:

$$P_2 c_2 + A_2 = (1+i)A_1 + Y_2, \tag{7}$$

where, as we just said, we must have $A_2 = 0$, and A_1 may be positive or negative.

This timing of events is depicted by the timeline in **Figure 20**, which is crucial to understand.

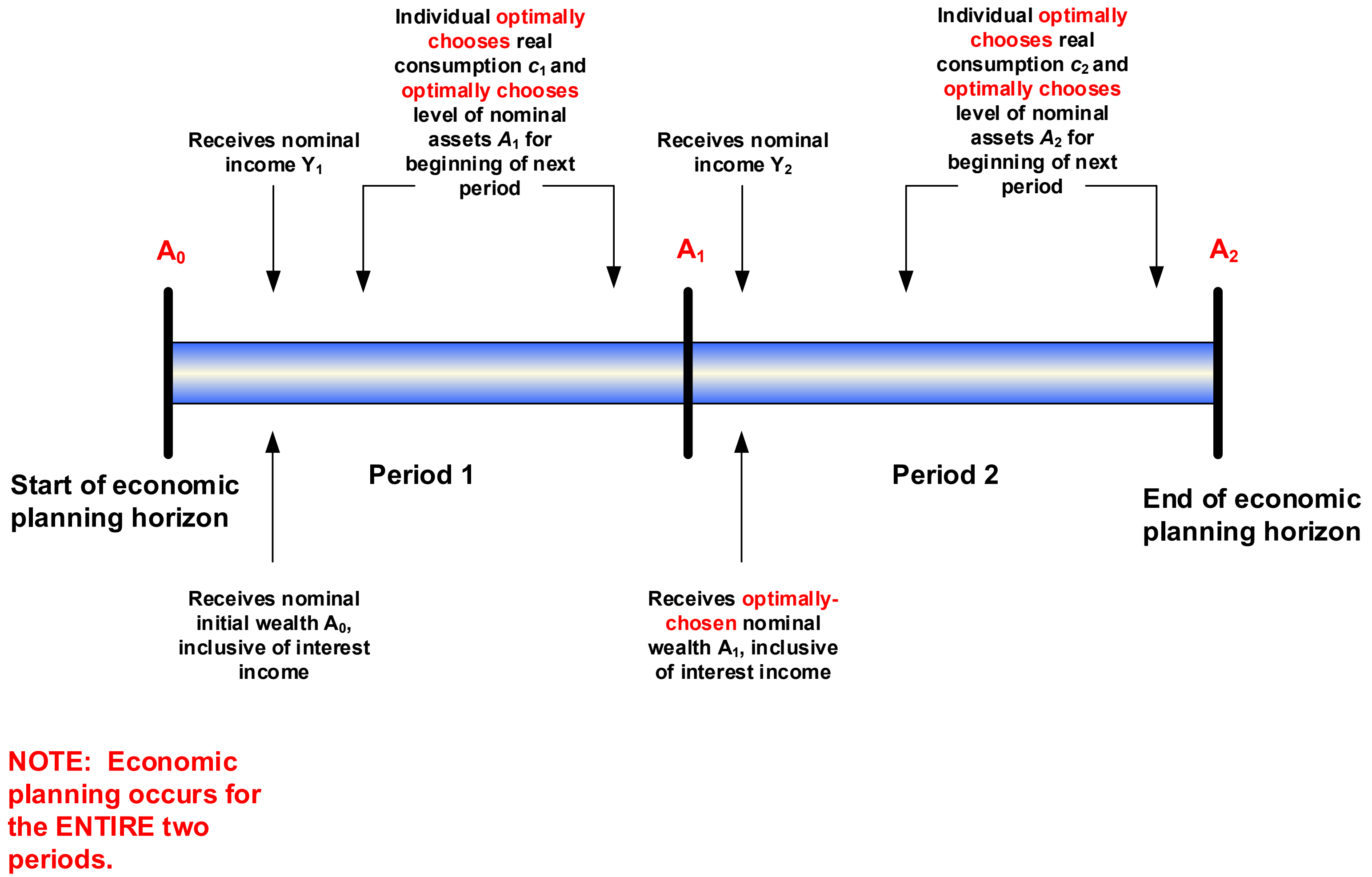

Figure 20. Timing of events in the two-period consumption framework, stated in nominal units.

Before making our next point, we introduce important new terminology. We define an individual's **private savings in a given time period as the difference between his total income in that period and his total expenditures in that period.** The two main categories of expenditures for individuals in any economy are consumption and taxes. We have not yet discussed taxes, but we will soon. Examining the period-1 budget constraint (6) above, we see that the individual's total income in period 1 is $iA_0 + Y_1$ (the sum of his labor income and interest income), and his total expenditure on consumption in period 1 is P_1c_1. Thus, we have that the individual's private savings in period 1 is

$$S_1^{priv} = iA_0 + Y_1 - P_1c_1, \tag{8}$$

where the "priv" superscript indicates that this is the savings of the private individual.[28] If we rearrange expression (6) a bit, we get that

$$A_1 - A_0 = iA_0 + Y_1 - P_1c_1. \tag{9}$$

Comparing expressions (8) and (9), we see that $S_1^{priv} = A_1 - A_0$.

Thus, the private individual's savings in period 1 is equal to the change in his wealth during period 1. This is a second useful way of computing an individual's private savings – as the change in wealth. To re-emphasize, this is a CRITICAL idea to understand, as it is pervasive throughout macroeconomic analysis. At the end of the chapter, we will emphasize this point again with rigorous definitions of "stock variables" and "flow variables."

To continue the savings account example from above, starting from an initial balance of $1,000 if you withdrew $400 from your savings account during the course of one year (and made no deposits), your savings during the course of the year would be $600-$1000 = -$400. That is, you would have dissaved during the year.

Similarly, the private individual's savings in period 2 is $S_2^{priv} = iA_1 + Y_2 - P_2c_2$, which, using the period-2-budget constraint, can also be expressed as $S_2^{priv} = A_2 - A_1$.

Lifetime Budget Constraint

Examining the period-1 budget constraint and the period-2 budget constraint, we see that they are linked by wealth at the beginning of period 2, A_1. Mathematically, this is the only term that appears in both expressions. **The economic interpretation, an important one, is that an individual's wealth position is what links economic decisions of the past with economic decisions of the future.** Again continuing the savings account

[28] Later, we will also have something called "public savings," in which the government engages – we will denote this by S^{gov}.

example from above, the $1,000 in your savings account somehow reflects your past income and consumption decisions. Obviously, just knowing that you currently have $1,000 in your savings account does not allow anyone to know exactly what or how much "stuff" you bought in the past or how much income you earned in the past. Nonetheless, it is essentially a summary of your past income and consumption behavior, albeit a condensed one. The fact that you have $1,000 in your account now implies some level of interest income for you in the upcoming year, income which is available for your consumption needs over the next year. Thus, that $1,000 is a reflection of your past economic behavior and represents part of your future economic opportunities.

Thus, economic decisions over time are linked by wealth. A useful first approximation to actual economic behavior is to suppose that individuals are completely rational over the course of their lifetimes in the sense that they save and/or borrow appropriately during their whole lifetimes. In the context of our two-period model here, such an assumption amounts to an individual deciding on his consumption and savings for his whole life (i.e., both period 1 and period 2) at the beginning of period 1. This latter point is an important one for the analysis of the two-period model: **all of our analysis of the two-period model proceeds from the point of view of the very beginning of period one.** That is, we will consider the very beginning of period one as the "moment in time" in which our (and the consumer's) analysis is conducted; hence, in our (and the consumer's) analysis of the two-period world, the entire two periods will always be yet to unfold.

Proceeding, then: armed with the assumption of rationality on the part of consumers and the perspective of economic events from the very beginning of period one, it is neither the period-1 budget constraint alone nor the period-2 budget constraint alone that is the relevant one for decision-making, but rather a combination of both of them.[29] The way to combine the budget constraints (6) and (7) is to exploit the observation that A_1 is the only term that appears in both. The mathematical strategy to employ is to solve for A_1 from one of the constraints and then substitute the resulting expression into the other constraint. Doing this will yield the individual's **lifetime budget constraint** – which we will abbreviate **LBC** for short.

Let us proceed by first solving for A_1 in expression (7). After a couple of steps of algebra, we get

$$A_1 = \frac{P_2 c_2}{(1+i)} - \frac{Y_2}{(1+i)}, \tag{10}$$

where we have used the fact that $A_2 = 0$ from above.[30] Inserting this resulting expression for A_1 into the period-1 budget constraint in (6) above yields

[29] Keep this point in mind when we later formulate, two different types of Lagrange problems to analyze the two-period framework.

[30] It is a good idea for you to verify these algebraic manipulations and the ones that follow for yourself.

$$P_1 c_1 + \frac{P_2 c_2}{(1+i)} = Y_1 + \frac{Y_2}{(1+i)} + (1+i)A_0, \tag{11}$$

which is the LBC. The LBC has very important economic meaning. The right-hand-side of expression (11) represents the **present discounted value of lifetime resources,** which takes into account both initial wealth as well as all lifetime labor income.[31] The left-hand-side of expression (11) represents the **present discounted value of lifetime consumption,** which takes into account consumption in all periods of the individual's life (here, only two periods). Thus, over the course of his lifetime, the individual spends all his lifetime resources on lifetime consumption, leaving nothing behind when he dies (and indeed why should he because, after all, the world ends with certainty at the end of period 2). It is this LBC that our perfectly rational individual uses in making his choices over time. As such, in order to proceed graphically, we need to represent this LBC in $c_1 - c_2$ space.

Before graphing the LBC, we make one simplifying assumption, that $A_0 = 0$, which means the individual begins his economic life with zero initial wealth (and zero initial debts). None of the qualitative results change if we do not make this assumption – it simply makes the graphical analysis to follow more straightforward.

To graph the LBC with c_2 on the vertical axis and c_1 on the horizontal axis, we need to solve expression (11) for c_2, which gives us, after a few lines of algebra,

$$c_2 = -\left(\frac{P_1(1+i)}{P_2}\right)c_1 + \left(\frac{1+i}{P_2}\right)Y_1 + \frac{Y_2}{P_2}. \tag{12}$$

Thus, the vertical intercept is the entire term $\left(\frac{1+i}{P_2}\right)Y_1 + \frac{Y_2}{P_2}$, and the slope is the term $-\left(\frac{P_1(1+i)}{P_2}\right)$. The graph of the LBC is in **Figure 21**.

[31] You should be familiar with the notion of present discounted value from introductory economics – if your recollection is a bit hazy on this point, now is the time to refresh yourself because we will use the concept repeatedly.

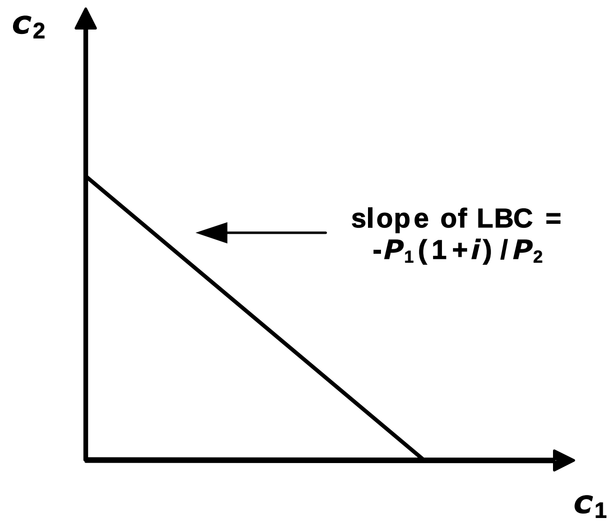

Figure 21. The lifetime budget constraint (LBC) of the individual, with the simplifying assumption that $A_0 = 0$.

Optimal Intertemporal Choice – Consumption and Savings

As in all of consumer theory, the individual's actual optimal choice is determined by the interaction of his budget constraint and his indifference map (i.e., his utility function) – the former represents all of the choices available to him and the latter represents his own personal preferences. **Figure 22** depicts an example, in which the individual's optimal choice is c_1^* in period 1 and c_2^* in period 2.

Also shown in **Figure 22** are the individual's labor incomes in both period 1 and period 2. Actually, what are shown are Y_1/P_1 and Y_2/P_2, which represent **real labor income** in the two periods, respectively. We will soon discuss exactly what is meant by this term, but for now just think of it as the labor income we have been discussing all along in this two-period model. We see in **Figure 22** that consumption c_1^* in period 1 is higher than real labor income in period 1 Y_1/P_1. This individual is spending more in period 1 than he earns, which means that the individual must be decumulating wealth (i.e., borrowing) during period 1. We can see this mathematically by looking at the period 1 budget constraint in expression (6) (and recall our simplifying assumption that $A_0 = 0$). Rearranging that expression a bit gives

$$c_1 - \frac{Y_1}{P_1} = -\frac{A_1}{P_1}. \tag{13}$$

So for the individual in **Figure 22**, the left-hand-side of expression (13) is positive, which must mean that A_1 for this individual is negative. This individual is in debt at the end of period 1. By similar logic and using the period 2 budget constraint in expression (7) we have that

$$c_2 - \frac{Y_2}{P_2} = \frac{(1+i)A_1}{P_2}. \tag{14}$$

We already know that A_1 is negative, implying the left-hand-side of expression (14) must be negative, which is in fact the case looking at **Figure 22**. The reason why consumption is smaller than income in period 2 is because the individual has to repay the loan obligations he took on during period 1. Thus, consumption higher than labor income in one period has to be balanced with consumption lower than income in another period, a result which should strike you as not surprising.

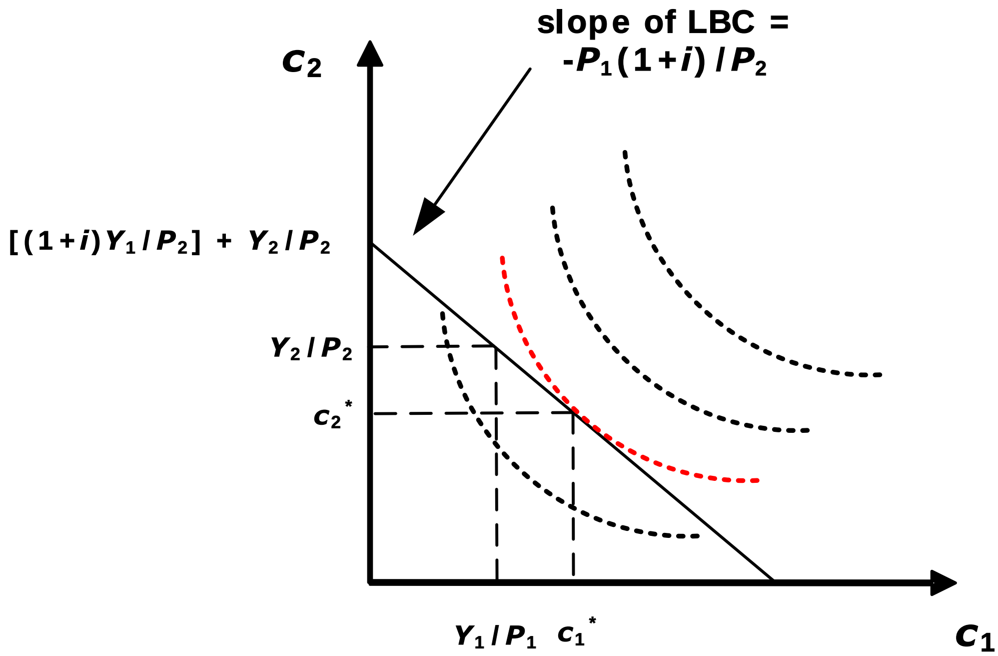

Figure 22. The interaction of the individual's LBC and his/her preferences (represented by the indifference map) determine the individual's optimal consumption over time, here c_1* in period 1 and c_2* in period 2.

One final point regarding the example in **Figure 22**: notice that no mention was made of interest income, only labor income – despite the careful distinction we made earlier between labor income and interest income. The reason for this is that when considering the **lifetime** choices he makes and as long as asset markets are perfectly functioning (we will discuss in more depth the content of this qualifier), the individual can completely disregard interest income because the only reason for the existence of non-zero wealth at the end of any period is simply to transfer resources across time.

When explicitly considering the lifetime decisions of an individual, as we are here, **those "intermediate" wealth positions appear to "completely cancel out."** Specifically, notice that **from a *mathematical* point of view,** A_1 does not appear at all in the LBC in expression (12), and the only relevant income for the individual is that which he receives in period 1 and period 2. **However, from an *economic* perspective, the A_1 net wealth term that links activities across time periods is still present. This is a *critical* point to understand about multi-period economic frameworks – there is some "state of economic conditions" that occurred in the past and have implications for current and future outcomes.**

Stocks vs. Flows

Understanding the two-period model (and as Figure 20 portrays) requires understanding a **critical** conceptual difference between two different types of variables: **stock variables** and **flow variables.** This conceptual difference arises **entirely because of the dynamic nature of the two-period framework.**

Stock Variables (alternative terminology: Accumulation Variables)

Quantity variables whose natural measurement occurs at a particular moment in time

Examples:

- **Checking account balance**
- **Credit card indebtedness**
- **Mortgage loan payoff**
- **College loan balance**

In our two-period model so far, and as displayed in Figure 20, the three stock variables (aka accumulation variables) are A_0, A_1, and A_2.

Flow Variables

Quantity variables whose natural measurement occurs during the course of a given interval of time

Examples:

- **Income**
- **Consumption**
- **Savings**

In our two-period model so far, and as displayed in Figure 20, the six flow variables are c_1, c_2, Y_1, Y_2, S_1, and S_2.
