Chapter -1
Mathematical Refresher

The concept of a function is a very general and powerful one. A function is a mathematical object that serves as a fundamental tool in many fields of analysis. We will not here give a rigorous or comprehensive treatment of the mathematical notion of a function. The purpose here is to (re)familiarize you with the basic concepts and the most important ways in which we will use functions as we develop tools of economic analysis in this course.

Abstract Functions and Functional Forms

A **function** transforms an **input** into an **output.** More specifically, a function is a rule that specifies how an input is to be transformed into some output. At its simplest level, the level with which we will be concerned, the input and outputs will all be numbers. In general, any function can have multiple inputs and multiple outputs. Every function that we will use will have only one output – that is, a function whose operation results in only one numeric value as its output. However, we will regularly encounter functions that have multiple inputs, in addition to functions that have simply a single input.

A function can be written and used in abstract form, as when we simply write and use the function $f(x)$ without specifying anything further about what the function actually does. Often, however, in order to do something useful with a function, we need to specify a particular **functional form –** that is, we often need to specify what a function actually does (i.e., what the rule is). Some examples of common functional forms will help illustrate the concept:

$$f(x) = x^2 \tag{0.1}$$
$$f(x) = 2x + 8 \tag{0.2}$$
$$f(x) = \sqrt{x} \tag{0.3}$$
$$f(x) = \ln(x) \tag{0.4}$$
$$f(x, y) = \ln(x) + 0.8\ln(y) \tag{0.5}$$

In the above simple functions (functional forms), note that each function returns only one number as its output (as promised). Also note that function (0.5) is a function of two inputs, while the others are all functions of one input.

Arguments of Functions

To be a bit more formal mathematically, an input(s) to a function is commonly known as its **argument(s),** and the output of a function is commonly known as its **result** or **value.**

Using the functions defined above, we see that each of the functions (0.1) through (0.4) takes one argument named x. Function (0.5) takes two arguments named x and y.

When actually performing numerical calculations using functions, the x in each case of the functions (0.1) through (0.4) would be replaced with an actual number because it is meaningless to *square the letter x,* because only *numbers* can be squared. The leads to the distinction between **formal arguments** and **actual arguments.**

Think of a formal argument as a placeholder in an abstract function. In each function (0.1) through (0.4), the formal argument is x. In function (0.5), the two formal arguments are x and y. More will be said about replacing formal arguments with actual arguments, but first let's examine the components of a function definition.

Dissecting the Components of a Function Definition

Examining the components of a function definition will help illuminate what a function actually represents. Consider the simple function given in (0.1) above:

$$f(x) = x^2.$$

There is much to understand about this function definition. Proceeding left to right:

- The name of the function is f. There is nothing particularly special about the name given to a function -- f is a popular choice when trying to be as abstract as possible. Sometimes, the letter used to name a function is chosen so that it somehow represents a memorable aspect of the function. For example, the **m**oney **d**emand function in maroeconomics is often named M^D. But any name is perfectly valid. In the example under consideration, we could have written $g(x) = x^2$ or $F(x) = x^2$ or $h(x) = x^2$ or $ExampleFunction(x) = x^2$. In short, we could have given *any* name to the function, not only f.

- The parentheses ( ) contain the formal argument(s) of the function. In this case, the formal argument to the function f is x. In a function such as (0.5) above, the parentheses contain two arguments. An important point to note, similar to the point immediately above, is that the name of the formal argument is unimportant. In the example $f(x) = x^2$, the name of the formal argument is x. But it could have just as easily been named y, in which case the function definition would be $f(y) = y^2$. It could also have just as easily been named $\arg ument$, in which case the function definition would be $f(\arg ument) = \arg ument^2$. There would be absolutely no material change to the function definition if this were the case –

precisely because the formal argument is simply a placeholder and does not itself mean anything.

- One the right-hand-side of the equals sign is the **body** of the function. The body uses the formal argument(s) of the function and specifies what calculation should be performed. In our simple example above, the body specifies that the result should be the square of the argument. In function (0.2) above, the body specifies that the return value of the function should be two times the argument plus eight. Similarly for the other functions above.

Replacing Formal Arguments with Actual Arguments

As alluded to above, the usefulness of a function is in its ability to substitute actual numeric values for the formal arguments of the functions and thereby generate numeric results. **Table 1** computes the results of two simple inputs for two of our example functions. All that has been done is to substitute actual arguments (10 and 20 in these particular cases) for the formal arguments x in functions (0.1) and (0.2). Specifically, what has been done is that the actual arguments have been substituted for the formal arguments in the *body* of the functions. The body of the function is then numerically computed, and the resulting numeric value is the return value of the function.

Functional Form	**Input**	**Calculation**	**Output/Value**
$f(x) = x^2$	**10**	$\mathbf{10^2}$	**100**
$f(x) = x^2$	**20**	$\mathbf{20^2}$	**400**
$f(x) = 2x + 8$	**10**	**2(10) + 8**	**28**
$f(x) = 2x + 8$	**20**	**2(20) + 8**	**48**

Table 1

Note that in the absence of specifying actual arguments, the return value of the function is simply the body of the function itself – which *includes* the formal arguments.

Using Abstract Functions in Algebraic Manipulations

A very important concept to understand is that functions can be manipulated algebraically just as "ordinary" variables and numbers are manipulated algebraically. The following visual illustrates this concept:

$x+7=12$ $\qquad$ $f(x)+7=12$

$\Downarrow$ **subtract 7 from both sides** $\qquad$ $\Downarrow$ **subtract 7 from both sides**

$x=5$ $\qquad$ $f(x)=5$

In the simple expression $x+7=12$, in order to solve for x, the value 7 is subtracted from both sides of the equality, which yields the solution $x=5$. Completely analogously, if the expression $f(x)+7=12$ is to be solved for $f(x)$, simply subtract 7 from both sides of the equality, which yields the solution $f(x)=5$. If a particular functional form for f is not specified, then this is as far as we can take the calculation. That is, when no functional form is given, $f(x)=5$ is a perfectly valid solution!

However, if a functional form *is* specified, then we can proceed a bit further. Continuing with our example from the preceding paragraph, if the function specified were $f(x)=2x+8$, then we can solve for x as follows:

$$f(x)=5$$

$\Downarrow$ **replace $f(x)$ by the given functional form**

$$2x+8$$

$\Downarrow$ **solve for x**

$$x=-3/2$$

Trying for yourself the other functional forms we have encountered would be a good exercise at this point.

The main point to understand is that performing algebraic manipulations with (abstract or particular) functions is just like performing algebraic manipulations with "ordinary" variables and numbers. There is nothing mysterious here, and you should make yourself comfortable with this concept and its mechanics because we will use it repeatedly throughout our study.

Key Concepts

- A function takes (numeric) inputs and results in (numeric) outputs.

- When provided with a specific functional form for a function, computations can be carried further then if no functional form is specified.

- When performing numerical calculations, if actual arguments are provided, the actual arguments replace the formal argument in the body of the function definition.

- Abstract functions can be manipulated algebraically just like ordinary variables and numbers.

Lagrange Optimization

With the concept of a function in hand, we now provide a brief overview of **constrained optimization.** A constrained optimization problem is one in which the goal is to find numerical values for the arguments of a function in such a way that the numerical value of that function is maximized (or minimized) **and** that satisfy some pre-specified relationship(s) between the arguments being chosen.

Many of our economic applications of constrained optimization will involve functions of two arguments, so we first illustrate the method of **Lagrange optimization,** which is the standard mathematical tool used to solve constrained optimization problems, using a problem with two variables. Note, however, that the Lagrange method readily applies to functions of one, three, four, or any number of variables.

Consider the following mathematical constrained optimization problem. There is a function $f(x, y)$, and the goal is to find the numerical values of x and y that, when used simultaneously in f, maximize the numerical value of f **and** satisfy the relationship $g(x, y) = 0$. That is, g is some other function that the two variables x and y satisfy – but the goal is not to maximize (or minimize) g, the goal is to maximize f.

The Lagrange method in this problem proceeds as follows. Define an auxiliary variable λ (the Greek letter "lambda"). The variable λ is the **Lagrange multiplier.** With the Lagrange multiplier, construct the following function, called the **Lagrange function:**

$$L(x, y, \lambda) = f(x, y) + \lambda g(x, y).$$

That is, the Lagrange function L is a function of **three** variables: x, y, and the newly-constructed variable λ. The Lagrange function is made up of two components summed together: the **objective function** f which is to be maximized and λ times the **constraint function** g.

The next step in the procedure is to compute the partial derivatives of L with respect to each of its three arguments and set each resulting expression to zero. Using general notation, these three expressions are:

$$\frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0$$
$$\frac{\partial L}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0$$
$$\frac{\partial L}{\partial \lambda} = g(x, y) = 0$$

These three equations are the **first-order conditions** of the optimization problem under consideration. These three expressions are three equations in the three unknowns, x, y, and λ, which typically can be solved for unique values of the three unknowns once we specify particular functional forms for the functions $f(x, y)$ and $g(x, y)$ (recall our discussion of functions above). Note that the third expression is in fact just the constraint on the optimization problem. This is a general principle: the first-order condition of the Lagrangian with respect to the Lagrange multiplier always delivers back the constraint function.

Remember the goal here is to ultimately to solve for x and y (and the Lagrange multiplier λ). We can solve each of the first two equations above for λ:

$$\lambda = -\frac{\partial f / \partial x}{\partial g / \partial x}$$
$$\lambda = -\frac{\partial f / \partial y}{\partial g / \partial y}$$

Setting these two equal to each other, we find that

$$\frac{\partial f / \partial x}{\partial g / \partial x} = \frac{\partial f / \partial y}{\partial g / \partial y},$$

Or, completely equivalently,

$$\frac{\partial f / \partial y}{\partial f / \partial x} = \frac{\partial g / \partial y}{\partial g / \partial x}.$$

This expression literally states that **at the optimal solution,** the ratio of partial derivatives of the objective function f is equal to the ratio of partial derivatives of the constraint function g. At this point, this is a completely abstract mathematical idea, but the basic result – that at the optimal solution, the ratio of partials of the objective function is equal to the ratio of partials of the constraint function – will be critical for many of the economic ideas we study, so it is well worth it to understand this idea as well as possible now.

The "optimality condition" (a term we will encounter in more precise instances soon) captured by the previous expression is one that **must** be satisfied **at the optimal solution.** Away from the optimal solution, however, this expression need not be satisfied (and in general will not be). Note well the content of these last two statements.

The multiplier λ has been eliminated from this last expression. This last expression coupled with the first-order condition of the Lagrangian with respect to λ, now comprise two equations in the two unknowns x and y. Given functional forms for f and g, we would be able to compute the required partial derivatives and thus solve for the optimal values of x and y (i.e., that combination of x and y that yields the maximum value of f and satisfies the constraint $g(x,y)=0$).

To take a concrete example to see how the Lagrange technique yields a solution, suppose $f(x,y)=\ln x+\ln y$ and $g(x,y)=x+y-5=0$. The necessary partial derivatives are: $\partial f/\partial x=1/x$, $\partial f/\partial y=1/y$, $\partial g/\partial x=1$, and $\partial g/\partial y=1$. With these partials, the optimality condition becomes

$$\frac{\partial f/\partial y}{\partial f/\partial x}=\frac{\partial g/\partial y}{\partial g/\partial x}\Rightarrow\frac{1/y}{1/x}=\frac{1}{1},$$

which easily simplifies to $x=y$. Thus, we now know that for this example, at the optimal solution (but not away from the optimal solution), $x=y$. Use this relationship in the constraint function (which, recall, is simply the first-order condition of the Lagrangian with respect to the multiplier), giving us $x+x-5=0$. Clearly, the solution is $x=2.5$, which then also implies that $y=x=2.5$. The optimization problem is now solved: the values of x and y that sum to 5 and maximize the given function $\ln x+\ln y$ are $x=2.5, y=2.5$.

We have illustrated the Lagrange method using one constraint function. The method readily generalizes to handle two, three, four, or any arbitrary number of constraints on a given optimization problem. We will encounter economic applications in which there are multiple constraint functions on an optimization problem. To start simply, consider an example in which there are two constraint functions, $g(x,y)=0$ as well as $h(x,y)=0$, that must be satisfied in the optimization of the function $f(x,y)$. In order to handle two constraints, we need two Lagrange multipliers – let's name them λ_1 and λ_2. The Lagrange function in this case would be

$$L(x,y,\lambda_1,\lambda_2)=f(x,y)+\lambda_1 g(x,y)+\lambda_2 h(x,y).$$

The Lagrange function L here is a function of the **four** variables x, y, λ_1, and λ_2, and we must compute the partial derivatives of L with respect to each of its four arguments and set each resulting expression to zero. Again using general notation, the four first-order conditions are

$$\frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} + \lambda_1 \frac{\partial g}{\partial x} + \lambda_2 \frac{\partial h}{\partial x} = 0$$
$$\frac{\partial L}{\partial y} = \frac{\partial f}{\partial y} + \lambda_1 \frac{\partial g}{\partial y} + \lambda_2 \frac{\partial h}{\partial y} = 0$$
$$\frac{\partial L}{\partial \lambda_1} = g(x, y) = 0$$
$$\frac{\partial L}{\partial \lambda_2} = h(x, y) = 0$$

which are four equations in four unknowns; once again, in general, the system of equations can be solved to yield a unique solution for each of the variables, although of course the algebra here is a bit more tedious because there are more equations to work through.

Implicit Function Theorem

Although we will use this concept sparingly, the implicit function theorem (IFT) is a clever way of obtaining a derivative of one argument in a function with respect to another argument of that function **in a way that maintains the output value.**

As a simple warm-up, suppose that $f(x, y) = x + y$. If $x = 5$ and $y = 3$, then obviously the output value is $f(x, y) = 8$. If we wanted to maintain the output value $f(x, y) = 8$ but want to change the **mix** between *x* and *y*, there are clearly an infinite number of combinations. One combination is $x = 3$ and $y = 5$. Another combination is $x = 4$ and $y = 4$. Yet another combination is $x = 1.235$ and $y = 6.765$. And so on. Thus, for every one unit change in the input argument *x*, there must be a one unit change in the argument *y* **in the equal and opposite direction** in order to maintain $f(x, y) = x + y = 8$.

More formally, given a function $f(x, y)$, the derivative of y (which, note, is one of the arguments of the *f* function) with respect to x (which, note, is also one of the arguments of the *f* function) is given by

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}.$$

To use the IFT in a more interesting example, suppose $f(x, y) = xy^2$. To compute the partial derivative of f with respect to x, we treat y as a constant, in which case we obtain $\partial f / \partial x = y^2$, and to compute the partial derivative of f with respect to y, we treat x as a constant, in which case we obtain $\partial f / \partial y = 2xy$. The IFT then tells us that

$$\frac{dy}{dx} = -\frac{y^2}{2xy}$$
$$= -\frac{y}{2x}$$

Thus, for every one unit change in the argument x, there must be change in the argument y of $-\frac{y}{2x}$ units in order to maintain a particular value of $f(x, y)$.

Elasticity

Very often important in economic analysis – so now we are moving away from abstract mathematics basics – is the sensitivity of one variable to a change in another variable. That is, when one variable changes, how much impact does it have on another variable. Note that elasticity is not the same concept as the implicit function theorem.

A classic example is the sensitivity of demand for a particular good when a change has occurred in its market price. Using this example of the sensitivity of quantity demanded, its elasticity is qualitatively defined as

$$\varepsilon_{q^d,p} = \frac{\text{\% change in quantity demanded of a good}}{\text{\% change in market price of that good}}.$$

The notation ε (the Greek letter "epsilon") is often used to describe elasticity. In this example, it is the elasticity of quantity demanded with respect to its price, hence the two subscripts q^d and p. Implicit in being able to compute an elasticity is that we already know the functional relationship between the two variables. In our example, consider it to be the **market demand function $q^d(p)$.**[1]

There are two major elasticity concepts in economics: the **arc elasticity** and the **point elasticity.** As you may recall from basic microeconomics, an arc elasticity averages between two potentially widely-varying points on the known functional relationship. If the gap between these two points turns out to be very small, the arc elasticity is effectively the same as the point elasticity. For macroeconomic purposes, because changes that occur are typically "small," the important one is the point elasticity. Thus, the point elasticity should be thought of as **the *percentage* by which one variable changes when a different variable changes by *one percent*, *starting from* a particular pair of those variables.**

[1] Based on what we described above, the name of the function is q^d, the argument of the function is p, and the body is left unspecified.

The **point elasticity** is mathematically defined as

$$\varepsilon_{q^d,p} = \frac{\partial \ln q^d(p^{known})}{\partial \ln p^{known}} = \frac{\partial q^d(p^{known})}{\partial p^{known}} \cdot \frac{p^{known}}{q^d(p^{known})}.$$

This expression understandably seems very complicated, but it is for the sake of clarity. Suppose we know, based on demand function, the starting pair $(p^{known}, q^d(p^{known}))$, which is one single point on the demand function. Obtaining the point elasticity then requires computing the derivative of quantity demand with respect to price, **evaluated at the point** $(p^{known}, q^d(p^{known}))$. Multiplying this by $\frac{p^{known}}{q^d(p^{known})}$ yields the point elasticity of quantity demanded around the starting pair.

An example illustrates this. Suppose $q^d(p) = p^{\psi}$ (ψ is the lowercase Greek letter "psi"). This implies $\frac{\partial q^d}{\partial p} = \psi p^{\psi-1}$, and hence the point elasticity, after several steps of algebra, is

$$\begin{aligned}
\varepsilon_{q^d,p} &= \frac{\partial \ln q^d}{\partial \ln p} = \frac{\partial q^d}{\partial p} \cdot \frac{p}{q^d} \\
&= \frac{\psi p^{\psi-1} \cdot p}{q^d} \\
&= \frac{\psi p^{\psi}}{q^d} \\
&= \frac{\psi p^{\psi}}{p^{\psi}} \\
&= \psi.
\end{aligned}$$

Notice that in the fourth step, the known functional relationship $q^d(p) = p^{\psi}$ was substituted in, which is perfectly valid to do.