LINEAR APPROXIMATION OF THE BASELINE RBC MODEL

JANUARY 8, 2020

LINEARIZATION

- **For** $f(x,y,z)=0$**, multivariable Taylor linear expansion around** $(\bar{x},\bar{y},\bar{z})$

$$f(x,y,z)\approx f(\bar{x},\bar{y},\bar{z})+f_x(\bar{x},\bar{y},\bar{z})(x-\bar{x})+f_y(\bar{x},\bar{y},\bar{z})(y-\bar{y})+f_z(\bar{x},\bar{y},\bar{z})(z-\bar{z})$$

(Illustrative example in scalars)

LINEARIZATION OF THE RBC MODEL

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(Illustrative example in scalars)

- **Four equations describe the dynamic solution to RBC model**
  - **Consumption-leisure efficiency condition**

$$-\frac{u_n(c_t,n_t)}{u_c(c_t,n_t)}=z_t m_n(k_t,n_t)$$

  - **Consumption-investment efficiency condition**

$$u_c(c_t,n_t)=\beta E_t\left[u_c(c_{t+1},n_{t+1})\left(1-\delta+z_{t+1}m_k(k_{t+1},n_{t+1})\right)\right]$$

  - **Aggregate resource constraint**

$$c_t+k_{t+1}-(1-\delta)k_t=z_t m(k_t,n_t)$$

  - **Law of motion for TFP**

$$\ln z_{t+1}=(1-\rho_z)\ln\bar{z}+\rho_z\ln z_t+\varepsilon_{t+1}^z$$

STEADY STATE

- Deterministic steady state the natural local point of approximation
- Shut down all shocks and set exogenous variables at their means
- The Idea: Let economy run for many (infinite) periods
    - Time eventually "doesn't matter" any more
    - Drop all time indices

$$-\frac{u_n(\bar{c},\bar{n})}{u_c(\bar{c},\bar{n})} = \bar{z}m_n(\bar{k},\bar{n})$$

$$u_c(\bar{c},\bar{n}) = \beta u_c(\bar{c},\bar{n})\left[m_k(\bar{k},\bar{n})+1-\delta\right]$$

$$\bar{c}+\delta\bar{k} = \bar{z}m(\bar{k},\bar{n})$$

$$\ln\bar{z} = (1-\rho_z)\ln\bar{z}+\rho_z\ln\bar{z} \Rightarrow \bar{z}=\bar{z} \quad \text{(a parameter of the model)}$$

- Given functional forms and parameter values, solve for (*c*, *n*, *k*)
    - The steady state of the model
    - Taylor expansion around this point

LINEARIZATION ALGORITHMS

- **Schmitt-Grohe and Uribe (2004 *JEDC*)**
  - **A perturbation algorithm**
    - **A class of methods used to find an approximate solution to a problem that cannot be solved exactly, by starting from the exact solution of a related problem**
    - **Applicable if the problem can be formulated by adding a “small” term to the description of the exactly-solvable problem**
  - **(Matlab code available through Columbia Dept. of Economics web site – DO NOT USE IN THIS CLASS)**

LINEARIZATION ALGORITHMS

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- **Uhlig (1999, chapter in *Computational Methods for the Study of Dynamic Economies*)**
  - **Uses a generalized eigen-decomposition**
    - **Typically implemented with Schur decomposition (Sims algorithm)**
  - **Matlab code available on Uhlig's site**

LINEARIZATION OF THE RBC MODEL

Define co-state vector and state vector

$$y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix} \qquad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$

LINEARIZATION OF THE RBC MODEL

Define co-state vector and state vector

$$y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix} \qquad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$

Order model's dynamic equations in a vector $\equiv f(y_{t+1}, y_t, x_{t+1}, x_t) = 0$

Consumption-leisure efficiency condition

Consumption-investment efficiency condition

Aggregate resource constraint

Law of motion for TFP

LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

1. **Differentiate** $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** y_{t+1}

First derivatives with respect to:

$$\begin{array}{l|c|c|} & c_{t+1} & n_{t+1} \\ \hline \text{Consumption-leisure efficiency condition} & & \\ \hline \text{Consumption-investment efficiency condition} & & \\ \hline \text{Aggregate resource constraint} & & \\ \hline \text{Law of motion for TFP} & & \\ \hline \end{array} = f_{y_{t+1}}$$

LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

2. Differentiate $f\left(y_{t+1}, y_t, x_{t+1}, x_t\right)$ **with respect to (elements of)** y_t

First derivatives with respect to:

$$\begin{array}{l cc} & c_t & n_t \\ \textbf{Consumption-leisure efficiency condition} & & \\ \textbf{Consumption-investment efficiency condition} & & \\ \textbf{Aggregate resource constraint} & & \\ \textbf{Law of motion for TFP} & & \end{array} = f_{y_t}$$

LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

3. Differentiate $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** x_{t+1}

First derivatives with respect to:

$$\begin{array}{l|c|c|} & k_{t+1} & z_{t+1} \\ \hline \text{Consumption-leisure efficiency condition} & & \\ \hline \text{Consumption-investment efficiency condition} & & \\ \hline \text{Aggregate resource constraint} & & \\ \hline \text{Law of motion for TFP} & & \\ \end{array} = f_{x_{t+1}}$$

LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

4. Differentiate $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** x_t

First derivatives with respect to:

$$\begin{array}{lc} & \begin{array}{cc} k_t & \quad z_t \end{array} \\ \begin{array}{l} \textbf{Consumption-leisure efficiency condition} \\ \textbf{Consumption-investment efficiency condition} \\ \textbf{Aggregate resource constraint} \\ \textbf{Law of motion for TFP} \end{array} & \left[\begin{array}{cc} \quad & \quad \\ \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{array}\right] \end{array} = f_{x_t}$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t)\right] = E_t\begin{bmatrix} f^1(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^2(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^3(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^4(y_{t+1}, y_t, x_{t+1}, x_t) \end{bmatrix}$$

Consumption-leisure efficiency condition

Consumption-investment efficiency condition

Aggregate resource constraint

Law of motion for TFP

LINEARIZATION OF THE RBC MODEL

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Conjecture equilibrium decision rules

Note: *g*(.) and *h*(.) are time invariant functions!

"Perturbation parameter": governs size of shocks

$$y_t = g(x_t,\sigma)$$

$$x_{t+1} = h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1}$$

Matrix of standard deviations of state variables

LINEARIZATION OF THE RBC MODEL

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Conjecture equilibrium decision rules

Note: *g*(.) and *h*(.) are time invariant functions!

"Perturbation parameter": governs size of shocks

$$y_t=g(x_t,\sigma)$$

$$x_{t+1}=h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1}$$

Matrix of standard deviations of state variables

Substitute decision rules into dynamic equations

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]=0$$

$$=E_t\left[f(g(x_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$

$$=E_t\left[f(g(h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$

$$\equiv F(x_t,\sigma)$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]=0$$

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$$=E_t\left[f(g(h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$

$$\equiv F(x_t,\sigma)$$

$$\Downarrow$$

$$F_x(x_t,\sigma)=0 \qquad F_\sigma(x_t,\sigma)=0$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t) \right] = 0$$

$$= E_t\left[f(g(x_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}, x_t \right]$$

$$= E_t\left[f(g(h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}, x_t \right]$$

$$\equiv F(x_t,\sigma)$$

$\Downarrow$ **Using chain rule and suppressing arguments**

$$F_x(x_t,\sigma) =$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t)\right] = 0$$

$$= E_t\left[f(g(x_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, x_t\right]$$

$$= E_t\left[f(g(h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, x_t\right]$$

$$\equiv F(x_t, \sigma)$$

$$\Downarrow$$

Using chain rule and suppressing arguments

$$F_x(x_t, \sigma) = f_{y_{t+1}} \cdot g_x \cdot h_x$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]=0$$

$$=E_t\left[f(g(x_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$

$$=E_t\left[f(g(h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$

$$\equiv F(x_t,\sigma)$$

$$\Downarrow$$

Using chain rule and suppressing arguments

$$F_x(x_t,\sigma)=f_{y_{t+1}}\cdot g_x\cdot h_x+f_{y_t}\cdot g_x$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$
\begin{aligned}
E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t) \right] = 0 \\
&= E_t\left[f(g(x_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}, x_t \right] \\
&= E_t\left[f(g(h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}, x_t \right] \\
&\equiv F(x_t,\sigma)
\end{aligned}
$$

$\Downarrow$

Using chain rule and suppressing arguments

$$
F_x(x_t,\sigma) = f_{y_{t+1}} \cdot g_x \cdot h_x + f_{y_t} \cdot g_x + f_{x_{t+1}} \cdot h_x
$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]=0$$

$$=E_t\left[f(g(x_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$

$$=E_t\left[f(g(h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$

$$\equiv F(x_t,\sigma)$$

$$\Downarrow$$

Using chain rule and suppressing arguments

$$F_x(x_t,\sigma)=f_{y_{t+1}}\cdot g_x\cdot h_x+f_{y_t}\cdot g_x+f_{x_{t+1}}\cdot h_x+f_{x_t}$$

$$=0$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]=0$$

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$$\equiv F(x_t,\sigma)$$

$$\Downarrow$$

Using chain rule and suppressing arguments

$$F_x(x_t,\sigma)=f_{y_{t+1}}\cdot g_x\cdot h_x+f_{y_t}\cdot g_x+f_{x_{t+1}}\cdot h_x+f_{x_t}$$

$$=0$$

Setting σ = 0 shuts down shocks

Holds, in particular, at the deterministic steady state $(\bar{x},0)$

$$F_x(\bar{x},0)=f_{y_{t+1}}\cdot g_x\cdot h_x+f_{y_t}\cdot g_x+f_{x_{t+1}}\cdot h_x+f_x=0$$

Each term evaluated at the steady state – as per Taylor theorem

LINEARIZATION OF THE RBC MODEL

- **A quadratic equation in the elements of g_x and h_x evaluated at the steady state**

$$F_x(\bar{x},0) = f_{y_{t+1}}(\bar{x},0)\cdot g_x(\bar{x},0)\cdot h_x(\bar{x},0) + f_{y_t}(\bar{x},0)\cdot g_x(\bar{x},0) + f_{x_{t+1}}(\bar{x},0)\cdot h_x(\bar{x},0) + f_{x_t}(\bar{x},0) = 0$$

- **Solve numerically for the elements of g_x and h_x (use `fsolve` OR conduct an eigenvalue decomposition in Matlab)**

LINEARIZATION OF THE RBC MODEL

- A quadratic equation in the elements of g_x and h_x evaluated at the steady state

$$F_x(\bar{x},0) = f_{y_{t+1}}(\bar{x},0)\cdot g_x(\bar{x},0)\cdot h_x(\bar{x},0) + f_{y_t}(\bar{x},0)\cdot g_x(\bar{x},0) + f_{x_{t+1}}(\bar{x},0)\cdot h_x(\bar{x},0) + f_{x_t}(\bar{x},0) = 0$$

- Solve numerically for the elements of g_x and h_x (use `fsolve` OR conduct an eigenvalue decomposition in Matlab)
- Recall conjectured equilibrium decision rules

$$y_t = g(x_t,\sigma)$$

$$x_{t+1} = h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}$$

LINEARIZATION OF THE RBC MODEL

- **A quadratic equation in the elements of g_x and h_x evaluated at the steady state**

$$F_x(\bar{x},0) = f_{y_{t+1}}(\bar{x},0)\cdot g_x(\bar{x},0)\cdot h_x(\bar{x},0) + f_{y_t}(\bar{x},0)\cdot g_x(\bar{x},0) + f_{x_{t+1}}(\bar{x},0)\cdot h_x(\bar{x},0) + f_{x_t}(\bar{x},0) = 0$$

- **Solve numerically for the elements of g_x and h_x (use `fsolve` OR conduct an eigenvalue decomposition in Matlab)**
- **Recall conjectured equilibrium decision rules**

$$y_t = g(x_t,\sigma)$$

$$x_{t+1} = h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}$$

- **First-order approximation is**

$$y_t = g(x_t,\sigma) \approx g(\bar{x},0) + g_x(\bar{x},0)(x_t - \bar{x}) + g_\sigma(\bar{x},0)\sigma$$

$$x_{t+1} = h(x_t,\sigma) \approx h(\bar{x},0) + h_x(\bar{x},0)(x_t - \bar{x}) + h_\sigma(\bar{x},0)\sigma$$

LINEARIZATION OF THE RBC MODEL

- **A quadratic equation in the elements of g_x and h_x evaluated at the steady state**

$$F_x(\bar{x},0) = f_{y_{t+1}}(\bar{x},0)\cdot g_x(\bar{x},0)\cdot h_x(\bar{x},0) + f_{y_t}(\bar{x},0)\cdot g_x(\bar{x},0) + f_{x_{t+1}}(\bar{x},0)\cdot h_x(\bar{x},0) + f_{x_t}(\bar{x},0) = 0$$

- **Solve numerically for the elements of g_x and h_x (use `fsolve` OR conduct an eigenvalue decomposition in Matlab)**
- **Recall conjectured equilibrium decision rules**

$$y_t = g(x_t,\sigma)$$

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- **First-order approximation is**

= 0

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$$x_{t+1} = h(x_t,\sigma) \approx h(\bar{x},0) + h_x(\bar{x},0)(x_t - \bar{x}) + h_\sigma(\bar{x},0)\sigma$$

= 0

SGU Theorem 1:

$g_\sigma = 0$ and $h_\sigma = 0$

LINEARIZATION OF THE RBC MODEL

- **A quadratic equation in the elements of g_x and h_x evaluated at the steady state**

$$F_x(\bar{x},0) = f_{y_{t+1}}(\bar{x},0)\cdot g_x(\bar{x},0)\cdot h_x(\bar{x},0) + f_{y_t}(\bar{x},0)\cdot g_x(\bar{x},0) + f_{x_{t+1}}(\bar{x},0)\cdot h_x(\bar{x},0) + f_{x_t}(\bar{x},0) = 0$$

- **Solve numerically for the elements of g_x and h_x (use `fsolve` OR conduct an eigenvalue decomposition in Matlab)**
- **Recall conjectured equilibrium decision rules**

$$y_t = g(x_t,\sigma)$$
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- **First-order approximation is**

= 0

$$y_t = g(x_t,\sigma) \approx g(\bar{x},0) + g_x(\bar{x},0)(x_t-\bar{x}) + g_\sigma(\bar{x},0)\sigma$$
$$x_{t+1} = h(x_t,\sigma) \approx h(\bar{x},0) + h_x(\bar{x},0)(x_t-\bar{x}) + h_\sigma(\bar{x},0)\sigma$$

= 0

SGU Theorem 1:

$g_\sigma = 0$ and $h_\sigma = 0$

- **DONE!!!**
  - **Now conduct impulse responses, tabulate business cycle moments, write paper**

CERTAINTY EQUIVALENCE

- **Displayed by a model if decision rules do not depend on the standard deviation of exogenous uncertainty – e.g., PRECAUTIONARY SAVINGS!**

- **For stochastic problems with quadratic objective function and linear constraints, the decision rules are identical to those of the nonstochastic problem**

CERTAINTY EQUIVALENCE

- **Displayed by a model if decision rules do not depend on the standard deviation of exogenous uncertainty – e.g., PRECAUTIONARY SAVINGS!**

- **For stochastic problems with quadratic objective function and linear constraints, the decision rules are identical to those of the nonstochastic problem**

- **Here, we have**

$$y_t = g(x_t, \sigma) \approx g(\bar{x}, 0) + g_x(\bar{x}, 0)(x_t - \bar{x}) + \overset{= 0}{g_\sigma}(\bar{x}, 0)\sigma$$

$$x_{t+1} = h(x_t, \sigma) \approx h(\bar{x}, 0) + h_x(\bar{x}, 0)(x_t - \bar{x}) + \underset{= 0}{h_\sigma}(\bar{x}, 0)\sigma$$

- **SGU Theorem 1: $g_\sigma = 0$ and $h_\sigma = 0$**
    - **First-order approximated decision rules do not depend on the size of the shocks, which is governed by σ**
    - **Not the same thing as "exact CE," but refer to it as CE**

LINEARIZING THE RBC MODEL

- **Assume** $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$

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- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$
- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ **)**

LINEARIZING THE RBC MODEL

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- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ **)**
- **Compute first row of matrix f_{yt+1}**

$$\begin{array}{l|cc} & c_{t+1} & n_{t+1} \\ \hline \text{Consumption-leisure efficiency condition} & & \\ \text{Consumption-investment efficiency condition} & & \\ \text{Aggregate resource constraint} & & \\ \text{Law of motion for TFP} & & \end{array}$$

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- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ **)**
- **Compute first row of matrix** ***f_{yt+1}***

$$\begin{array}{l c} & \begin{array}{cc} \boldsymbol{c_{t+1}} & \boldsymbol{n_{t+1}} \end{array} \\ \begin{array}{l} \textbf{Consumption-leisure efficiency condition} \\ \textbf{Consumption-investment efficiency condition} \\ \textbf{Aggregate resource constraint} \\ \textbf{Law of motion for TFP} \end{array} & \left[\begin{array}{c:c} \mathbf{O} & \\ \hdashline & \\ \hdashline & \\ \hdashline & \end{array}\right] \end{array}$$

LINEARIZING THE RBC MODEL

- **Assume** $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$
- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ **)**
- **Compute first row of matrix** ***f_{yt+1}***

	c_{t+1}	n_{t+1}
Consumption-leisure efficiency condition	0	0
Consumption-investment efficiency condition		
Aggregate resource constraint		
Law of motion for TFP		

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- **Compute first row of matrix** f_{yt}

$$\begin{array}{l|c|c}
 & \boldsymbol{c_t} & \boldsymbol{n_t} \\
\textbf{Consumption-leisure efficiency condition} & \dfrac{\psi}{n_t} & -\dfrac{\psi c_t}{n_t^2} + \alpha(1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha-1} \\
\hline
\textbf{Consumption-investment efficiency condition} & & \\
\hline
\textbf{Aggregate resource constraint} & & \\
\hline
\textbf{Law of motion for TFP} & &
\end{array}$$

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- **Compute first row of matrix f_{xt+1}**

$$\begin{array}{l|cc|} & k_{t+1} & z_{t+1} \\ \textbf{Consumption-leisure efficiency condition} & 0 & 0 \\ \hline \textbf{Consumption-investment efficiency condition} & & \\ \hline \textbf{Aggregate resource constraint} & & \\ \hline \textbf{Law of motion for TFP} & & \end{array}$$

(Partial) Example

LINEARIZING THE RBC MODEL

- **Assume** $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$
- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ **)**
- **Compute first row of matrix f_{xt}**

$$\begin{array}{l|c|c}
 & \boldsymbol{k_t} & \boldsymbol{z_t} \\
\textbf{Consumption-leisure efficiency condition} & -\alpha(1-\alpha)z_t\dfrac{k_t^{\alpha-1}}{n_t^{\alpha}} & -(1-\alpha)k_t^{\alpha}n_t^{-\alpha} \\
\hline
\textbf{Consumption-investment efficiency condition} & & \\
\hline
\textbf{Aggregate resource constraint} & & \\
\hline
\textbf{Law of motion for TFP} & & \\
\end{array}$$

LINEARIZING THE RBC MODEL

- **In deterministic steady state, the first rows of f_{yt+1}, f_{yt}, f_{xt+1}, f_{xt} are**

$$\begin{array}{lcc}
f_{yt+1} & 0 & 0 \\
f_{yt} & \dfrac{\psi}{\bar{n}} & -\dfrac{\psi\bar{c}}{\bar{n}^2}+\alpha(1-\alpha)\bar{z}\bar{k}^{\alpha}\bar{n}^{-\alpha-1} \\
f_{xt+1} & 0 & 0 \\
f_{xt} & -\alpha(1-\alpha)\bar{z}\dfrac{\bar{k}^{\alpha-1}}{\bar{n}^{\alpha}} & -(1-\alpha)\bar{k}^{\alpha}\bar{n}^{-\alpha}
\end{array}$$

LINEARIZING THE RBC MODEL

- **In deterministic steady state, the first rows of $f_{yt+1}, f_{yt}, f_{xt+1}, f_{xt}$ are**

$$
\begin{array}{lcc}
f_{yt+1} & 0 & 0 \\
f_{yt} & \dfrac{\psi}{\bar{n}} & -\dfrac{\psi\bar{c}}{\bar{n}^{2}}+\alpha(1-\alpha)\bar{z}\bar{k}^{\alpha}\bar{n}^{-\alpha-1} \\
f_{xt+1} & 0 & 0 \\
f_{xt} & -\alpha(1-\alpha)\bar{z}\,\dfrac{\bar{k}^{\alpha-1}}{\bar{n}^{\alpha}} & -(1-\alpha)\bar{k}^{\alpha}\bar{n}^{-\alpha}
\end{array}
$$

- **How to compute derivatives f_{yt+1}, f_{yt}, f_{xt+1}, f_{xt}?**
  - **By hand (feasible for small models)**
  - **Schmitt-Grohe and Uribe Matlab analytical routines**
  - **Your own Maple or Mathematica programs**
  - **MuPad**
  - **Dynare package**

CALIBRATION? SOLUTION PROCEDURE?

- **Solving for the steady state?**

- **Choosing parameter values?**

- **Next: example code**