Intertemporal Models: Basics of Dynamic Programming

September 10, 2013

Dynamic Programming

- Can we represent intertemporal problems **recursively?**
  - Rather than **sequentially**

- Benefits
  - Allows application of series of theorems/results that guarantee a **solution exists in the space of functions**
  - Allows application of series of theorems/results that help **find solution in the space of functions**
  - Computational algorithms require it – computers can't handle infinite-dimensional objects!

- Costs
  - May rule out some solutions to the original (sequential) problem
  - Requires (a lot?) more structure on the problem
  - Sometimes (often?) not obvious how to recast sequential problem as recursive problem

- Ljungqvist and Sargent (2004, p. 16)

"The art in applying recursive methods is to find a convenient definition of a state. It is often not obvious what the state is, or even whether a finite-dimensional state exists."

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  - Allows application of series of theorems/results that help **find solution in the space of functions**
  - Computational algorithms require it – computers can't handle infinite-dimensional objects!
- Costs
  - May rule out some solutions to the original (sequential) problem
  - Requires (a lot?) more structure on the problem
  - Sometimes (often?) not obvious how to recast sequential problem as recursive problem
- Start with deterministic case
  - (Fairly) straightforward
  - Stochastic case requires more structure

From Sequential to Recursive

- **Lagrangian of consumer problem, with planning horizon *T***

$$V^0(a_{-1}, r_0;.) \equiv \max_{\{c_t, a_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t \left[u(c_t) + \lambda_t \left(y_t + (1+r_{t-1})a_{t-1} - c_t - a_t \right) \right]$$

- **State variables of consumer problem at beginning of any period *s***
  - **a_{s-1} (accumulation variable) – the critical one b/c a_τ, $\tau \geq s$, are choices**
  - **r_s (price-taker)**
  - **A sufficient summary of the dynamic position of the environment in which the consumer operates**

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  - **a_{s-1} (accumulation variable) – the critical one b/c a_τ, $\tau \geq s$, are choices**
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  - **A sufficient summary of the dynamic position of the environment in which the consumer operates**

- **Define $V^0(a_{-1}, r_0;.)$ as value function starting from period zero**
  - **The maximized value of the constrained optimization problem**
  - **As function of period-zero parameters of the problem**

- **Goal: recast problem of finding optimal sequence $\{c_t, a_t\}_{t=0,1,2,\ldots T}$ into problem of finding functions $\{V^i(.)\}_{t=0,1,2,\ldots T}$**
  - **(Actually, find $V^i(.)$ along with two other functions)**

From Sequential to Recursive

- **Write out more explicitly**

$$V^0(a_{-1},r_0;.) \equiv \max_{\{c_0,a_0,c_t,a_t\}_{t=1}^{\infty}} \left\{ \begin{aligned} & u(c_0)+\lambda_0\left(y_0+(1+r_{-1})a_{-1}-c_0-a_0\right) \\ & \quad +\sum_{t=1}^{T}\beta^t\left[u(c_t)+\lambda_t\left(y_t+(1+r_{t-1})a_{t-1}-c_t-a_t\right)\right] \end{aligned} \right\}$$

Separate terms

$$V^0(a_{-1},r_0;.) \equiv \max_{c_0,a_0}\left\{u(c_0)+\lambda_0\left(y_0+(1+r_{-1})a_{-1}-c_0-a_0\right)\right\}$$

$$+\max_{c_0,a_0}\left\{\max_{\{c_t,a_t\}_{t=1}^{\infty}}\sum_{t=1}^{T}\beta^t\left[u(c_t)+\lambda_t\left(y_t+(1+r_{t-1})a_{t-1}-c_t-a_t\right)\right]\right\}$$

Note the max inside the max

Adjust β factors

FROM SEQUENTIAL TO RECURSIVE

$$V^0(a_{-1}, r_0;.) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c_0 - a_0 \right) \right\}$$

Adjust β factors →

$$+\beta \cdot \max_{c_0, a_0} \left\{ \max_{\{c_t, a_t\}_{t=1}^{\infty}} \sum_{t=1}^{T} \beta^{t-1} \left[u(c_t) + \lambda_t \left(y_t + (1+r_{t-1})a_{t-1} - c_t - a_t \right) \right] \right\}$$

The value resulting from optimal decisions starting from period 1.

<u>Bellman Principle of Optimality:</u>
optimal decisions in the initial period induce a future state, from which (future) decisions are optimal (Bellman, 1957)

From Sequential to Recursive

$$V^0(a_{-1}, r_0;.) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c_0 - a_0 \right) \right\}$$

Adjust β factors

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Recursive representation of consumer problem

$$V^0(a_{-1}, r_0;.) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c_0 - a_0 \right) + \beta \cdot V^1(a_0, r_1;.) \right\}$$

- **Bellman Equation**
- **Can analyze optimization problem for period zero…**
  - **…given Bellman Principle of Optimality holds**
  - **(But how do $V^0(.)$ and $V^1(.)$ relate to each other?)**

BELLMAN EQUATION

- **Bellman Equation**

$$V^0(a_{-1}, r_0;.) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c_0 - a_0 \right) + \beta \cdot V^1(a_0, r_1;.) \right\}$$

  - **Starting point for recursive analysis**
  - **Applicable to finite *T*-period or $T \rightarrow \infty$ problems**
  - **Construction requires identifying state variables of optimization problem**

- ***T*-period problem**
  - **Solution involves sequence of functions $V^0(.)$, $V^1(.)$, … , $V^{T-1}(.)$, $V^T(.)$**
  - **$V^i(.)$ functions in general will differ – reflecting time until end of planning horizon**
  - **E.g., maximized value starting from age = 60 different from maximized value starting from age = 30 (intuitively)**

- **Infinite-horizon problem**
  - **Deterministic case: $V(.) = V^i(.) = V^j(.)$ all *i,j***
  - **Always an infinity of periods left to go**

Stochastic case?

Requires more structure…

BELLMAN EQUATION

- **Bellman Equation (for $T \to \infty$)**

$$V(a_{-1}, r_0; .) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 \left(y_0 + (1 + r_{-1}) a_{-1} - c_0 - a_0 \right) + \beta \cdot V(a_0, r_1; .) \right\}$$

- **Use to characterize optimal decisions**
- **Period-0 FOCs**

c_0:

a_0:

BELLMAN EQUATION

- **Bellman Equation (for $T \rightarrow \infty$)**

$$V(a_{-1}, r_0;.) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c_0 - a_0 \right) + \beta \cdot V(a_0, r_1;.) \right\}$$

- **Use to characterize optimal decisions**
- **Period-0 FOCs**

c_0:

a_0:

Return to this …

- ***Suppose*** **optimal choice characterized by $c_0 = c(a_{-1};.)$, $a_0 = a(a_{-1};.)$ ($c(.)$ and $a(.)$ time-invariant functions in infinite-period problem)**
  - **Insert in value function (can now drop max operator)**

$$V(a_{-1}, r_0;.) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1;.)$$

  - **Now compute marginal**

Bellman Equation

- **Bellman Equation (for *T* → ∞)**

$$V(a_{-1}, r_0; .) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1 + r_{-1}) a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1; .)$$

  - **Now compute marginal (suppress *r* argument of *c*(.) and *a*(.) functions)**

$$V_1(a_{-1}, r_0; .) =$$

BELLMAN EQUATION

- **Bellman Equation (for $T \rightarrow \infty$)**

$$V(a_{-1},r_0;.) \equiv u(c(a_{-1})) + \lambda_0\left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1})\right) + \beta \cdot V(a(a_{-1}),r_1;.)$$

  - **Now compute marginal (suppress *r* argument of *c*(.) and *a*(.) functions)**

$$V_1(a_{-1},r_0;.) =$$

$$\Rightarrow \quad V_1(a_{-1},r_0;.) =$$

evaluate at period 1 →

$$V_1(a_0,r_1;.) =$$

Envelope Condition

- **Envelope Theorem**
  - **In computing first-order effects of changes in a problem's parameters on the maximized value, can ignore how optimal choices will adjust**
    - **Intuition: because already at a max (marginal costs = marginal benefits)**
  - **Need only consider the direct effect**

Note: envelope theorem has nothing to do with dynamic programming

Bellman Equation

- **Bellman Equation (for $T \rightarrow \infty$)**

$$V(a_{-1}, r_0; .) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1; .)$$

- **Use to characterize optimal decisions**

- **Period-0 FOCs, now evaluated using $c(a_{-1})$, $a(a_{-1})$**

c_0: $u'(c(a_{-1})) - \lambda_0 = 0$

a_0: $-\lambda_0 + \beta V_1(a_0(a_{-1}), r_1; .) = 0$

Env: $V_1(a(a_{-1}), r_1; .) = \lambda_1 (1 + r_0)$

$$\Rightarrow \quad u'(c(a_{-1})) = \beta (1 + r_0) u'(c(a_0))$$

- **Seems like usual Euler equation from sequential analysis (deterministic)…**

Deterministic – Recursive Analysis

- **Solution of infinite-horizon consumer problem (starting from date zero)…**
- **…is a consumption decision rule $c(a_{-1};.)$, asset decision rule $a(a_{-1};.)$, and value function $V(a_{-1};.)$ that satisfies**

  - **Bellman equation**

$$V(a_{-1},r_0;.) \equiv u(c(a_{-1})) + \lambda_0\left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1})\right) + \beta \cdot V(a(a_{-1}),r_1;.)$$

  - **Euler equation**

by envelope theorem

$$u'(c(a_{-1})) = \beta V_1(a(a_{-1}),r_1;.) \longleftrightarrow u'(c(a_{-1})) = \beta(1+r_0)u'(c(a_{-1}))$$

    - **which is the TVC in the limit $t \rightarrow \infty$:** $\lim_{t\to\infty} \beta^t u'(c(a_{t-1}^*)) \cdot a(a_{t-1}^*) = 0$

  - **Budget constraint**

$$y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) = 0$$

taking as given $\left(a_{-1}, r_0, r_{-1}\right)$

DETERMINISTIC – SEQUENTIAL ANALYSIS

- **Solution of infinite-horizon consumer problem (starting from date zero)…**
- **is a consumption and asset sequence $\left\{c_t^*, a_t^*\right\}_{t=0}^{\infty}$ that satisfies**
  - **Sequence of flow budget constraints**

$$c_t^* + a_t^* = y_t + (1+r_{t-1})a_{t-1}^*, \quad t = 0,1,2,\ldots$$

  - **Sequence of Euler equations**

$$u'(c_t^*) = \beta u'(c_{t+1}^*)(1+r_t), \quad t = 0,1,2,\ldots$$

    - **which is the TVC in the limit *t* → ∞:** $\lim_{t\to\infty} \beta^t u'(c_t^*)a_t^* = 0$

taking as given $\left(\left\{r_t, y_t\right\}_{t=0}^{\infty}, a_{-1}, r_{-1}\right)$

Does solution to recursive problem coincide with solution to sequential problem?

RECURSIVE VS. SEQUENTIAL ANALYSIS

- **Does solution to recursive problem coincide with solution to sequential problem?**
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RECURSIVE VS. SEQUENTIAL ANALYSIS

- **Does solution to recursive problem coincide with solution to sequential problem?**
- **Does solution to sequential problem coincide with solution to recursive problem?**

- **In constructing Bellman representation ($T \rightarrow \infty$ case), the imposition of time-invariant functions $c(a)$, $a(a)$ potentially limits the class of solutions**
  - **In original sequential formulation, this is neither explicitly nor implicitly a requirement of the solution!**
- **In general (here without proof…)**
  - **Solution to the sequential problem is also a solution to the recursive problem**
  - **Solution to the recursive problem is also a solution to the sequential problem provided some further regularity conditions hold**
- **Stokey, Lucas, Prescott text (1989)**

Recursive vs. Sequential Analysis

- So why go recursive?
  - Allows application of series of theorems/results that guarantee a solution exists in the space of functions
  - Allows application of series of theorems/results that help find solution in the space of functions

Underlying theory:
Contraction Mapping Theorem, Blackwell's Sufficient Conditions for a Contraction, Theorem of the Maximum

THEORY

- **Blackwell's Sufficient Conditions for a Contraction: Let *X* be a subset of R^l and let *B*(*X*) be the set of bounded functions *f* : *X* → *R* with the sup norm. Let *T* : *B*(*X*) → *B*(*X*) be an operator satisfying**

  a. **(Monotonicity) *f*,*g* 2 *B*(*X*) and *f*(*x*) · *g*(*x*), for all *x* 2 *X*, implies (*Tf*)(*x*) · (*Tg*)(*x*), for all *x* 2 *X***

  b. **(Discounting) There exists some *β* 2 (0,1) such that**

  [*T*(*f*+*a*)](*x*) · (*Tf*)(*x*) + *βa*, for all *f* 2 *B*(*X*), *a* ¸ 0, *x* 2 *X*

  Then *T* is a contraction with modulus *β*.

  (Note: (*f*+*a*)(*x*) is the function defined by (*f*+*a*)(*x*) = *f*(*x*) + *a*)

THEORY

- Let (S, ρ) be a metric space and $T : S \rightarrow S$ be a function mapping set S into itself. T is a **contraction mapping (with modulus β)** if for some β 2 (0,1), $\rho(Tx, Ty) \cdot \beta\rho(x, y)$ for all x, y 2 S.

  Example: S = [a, b] with $\rho(x, y) = |x - y|$ (Euclidean norm)

- **Contraction Mapping Theorem:** If (S, ρ) is a metric space and $T : S \rightarrow S$ is a contraction mapping with modulus β, then
  a. T has exactly one fixed point v in set S.
  b. For any v_0 2 S, $\rho(T^n v_0, v) \cdot \beta\rho(x, y)$ for n = 0, 1, 2, …

- CMT states that a contraction mapping has a **unique fixed point,** and the fixed point can be found by iterative application of the mapping T starting starting from any point in S.

THEORY

- **General class of problems to which our (usual) economic optimization problems belong have the form**

  $(Tv)(x) = \sup_{y \in \Gamma(x)} [F(x,y) + \beta v(y)]$

- **For our economic theory: would like operator *T* to map the space *C*(*X*) of bounded continuous functions of the state vector into itself. Would also like to be able to characterize the set of maximizing values of *y* given *x*.**

- **Theorem of the Maximum: Let *X* be a subset of R^l, *Y* be a subset of R^m, let $f : X \times Y \rightarrow R$ be a (single-valued) continuous function, and let $\Gamma : X \rightarrow Y$ be a compact-valued and continuous correspondence. The problem we are interested in is of the form $\sup_{y \in \Gamma(x)} f(x,y)$. Then**

  a. ***sup*** **can be replaced with *max* because, for each *x*, the maximum is attained and the function $h(x) = \max_{y \in \Gamma(x)} f(x,y)$ is well defined and continuous**

  b. **The correspondence $G(x) = y \in \Gamma(x) : f(x,y) = h(x)$ is well defined, is non-empty, is compact-valued, and upper hemi-continuous.**

- **Theorem of the Maximum establishes the existence of the maximum of the problem.**

THEORY

- **Suppose in addition to the hypotheses of the Theorem of the Maximum, the correspondence *Γ* is convex-valued and the function *f* is strictly concave in *y*.**

  → Then *G* is single-valued. Call this function *g*, and *g* is continuous.

- **Establishes that, given these conditions and given the unique solution of the Bellman Equation, there is a unique *g* that is the optimal "decision rule."**

- **If $\{f_n(x,y)\}$ is a sequence of continuous functions converging to $f(x,y)$, each strictly concave in *y*, then the sequence of functions $\{g_n(x)\}$ (which are the argmax of the sequence $\{f_n(x,y)\}$) converges pointwise to $g(x)$, which is the argmax of $f(x,y)$.**

- **The latter result is very useful considered in the context of the Contraction Mapping Theorem. It guarantees that the solutions to the sequence of problems converges to the true solution.**

Recursive vs. Sequential Analysis

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Underlying theory:
Contraction Mapping Theorem, Blackwell's Sufficient Conditions for a Contraction, Theorem of the Maximum

  - Computational algorithms require it – computers can't handle infinite-dimensional objects!
    - Soon: simple computational algorithms

- Can't really "choose" whether want to analyze problem sequentially or recursively
  - All but the most limited of problems/models require computational solution
  - In which case model analysis is recursive

- What about stochastic dynamic programming?
  - Even more structure required....

STOCHASTIC DYNAMIC PROGRAMMING

- Even more structure required on the problem to recursively solve dynamic stochastic optimization problems

- **Main (new) technical problem**
  - **Branching** of event tree at each of T periods (possibly $T \rightarrow \infty$)

- **Main technical solution / assumption**
  - Assume risk follows **Markov** process
  - Which enables series of theorems / results from deterministic dynamic programming to work in stochastic case…
  - …given further technical regularity assumptions