INTERTEMPORAL MODELS: BASICS OF DYNAMIC PROGRAMMING

JANUARY 25, 2012

DYNAMIC PROGRAMMING

- Can we represent intertemporal problems recursively?
  - Rather than sequentially
- Benefits
  - Allows application of series of theorems/results that guarantee a solution exists in the space of functions
  - Allows application of series of theorems/results that help find solution in the space of functions
  - Computational algorithms require it – computers can't handle infinite-dimensional objects!
- Costs
  - May rule out some solutions to the original (sequential) problem
  - Requires (a lot?) more structure on the problem
  - Sometimes (often?) not obvious how to recast sequential problem as recursive problem
- Ljungqvist and Sargent (2004, p. 16)

  "The art in applying recursive methods is to find a convenient definition of a state. It is often not obvious what the state is, or even whether a finite-dimensional state exists."

Dynamic Programming

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  - Computational algorithms require it – computers can't handle infinite-dimensional objects!
- Costs
  - May rule out some solutions to the original (sequential) problem
  - Requires (a lot?) more structure on the problem
  - Sometimes (often?) not obvious how to recast sequential problem as recursive problem
- Start with deterministic case
  - (Fairly) straightforward
  - Stochastic case requires more structure

From Sequential to Recursive

- Lagrangian of consumer problem, with planning horizon *T*

$$V^0(a_{-1}, r_0; .) \equiv \max_{\{c_t, a_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t \left[u(c_t) + \lambda_t \left(y_t + (1+r_{t-1})a_{t-1} - c_t - a_t \right) \right]$$

- State variables of consumer problem at beginning of any period *s*
  - a_{s-1} (accumulation variable) – the critical one b/c a_τ, $\tau \geq s$, are choices
  - r_s (price-taker)
  - A sufficient summary of the dynamic position of the environment in which the consumer operates

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- **State variables of consumer problem at beginning of any period *s***
  - **a_{s-1} (accumulation variable) – the critical one b/c a_τ, $\tau \geq s$, are choices**
  - **r_s (price-taker)**
  - **A sufficient summary of the dynamic position of the environment in which the consumer operates**

- **Define $V^0(a_{-1}, r_0; .)$ as value function starting from period zero**
  - **The maximized value of the constrained optimization problem**
  - **As function of period-zero parameters of the problem**

- **Goal: recast problem of finding optimal sequence $\{c_t, a_t\}_{t=0,1,2,\ldots T}$ into problem of finding functions $\{V^t(.)\}_{t=0,1,2,\ldots T}$**
  - **(Actually, find $V^t(.)$ along with two other functions)**

From Sequential to Recursive

- **Write out more explicitly**

$$V^0(a_{-1}, r_0; .) \equiv \max_{\{c_0, a_0, c_t, a_t\}_{t=1}^{\infty}} \left\{ \begin{aligned} & u(c_0) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c_0 - a_0 \right) \\ & \quad + \sum_{t=1}^{T} \beta^t \left[u(c_t) + \lambda_t \left(y_t + (1+r_{t-1})a_{t-1} - c_t - a_t \right) \right] \end{aligned} \right\}$$

From Sequential to Recursive

- Write out more explicitly

$$V^0(a_{-1},r_0;.) \equiv \max_{\{c_0,a_0,c_t,a_t\}_{t=1}^{\infty}} \left\{ \begin{array}{l} u(c_0)+\lambda_0\left(y_0+(1+r_{-1})a_{-1}-c_0-a_0\right) \\ \qquad +\sum_{t=1}^{T}\beta^t\left[u(c_t)+\lambda_t\left(y_t+(1+r_{t-1})a_{t-1}-c_t-a_t\right)\right] \end{array} \right\}$$

↓ **Separate terms**

$$V^0(a_{-1},r_0;.) \equiv \max_{c_0,a_0}\left\{u(c_0)+\lambda_0\left(y_0+(1+r_{-1})a_{-1}-c_0-a_0\right)\right\}$$

$$+\max_{c_0,a_0}\left\{\max_{\{c_t,a_t\}_{t=1}^{\infty}}\sum_{t=1}^{T}\beta^t\left[u(c_t)+\lambda_t\left(y_t+(1+r_{t-1})a_{t-1}-c_t-a_t\right)\right]\right\}$$

Note the max inside the max

↓ **Adjust β factors**

From Sequential to Recursive

Adjust β factors →

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$$+\beta\cdot\max_{c_0,a_0}\left\{\underbrace{\max_{\{c_t,a_t\}_{t=1}^{\infty}}\sum_{t=1}^{T}\beta^{t-1}\left[u(c_t)+\lambda_t\left(y_t+(1+r_{t-1})a_{t-1}-c_t-a_t\right)\right]}\right\}$$

From Sequential to Recursive

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$V^0(a_{-1},r_0;.)$ is value function starting from period 0.

Bellman Principle of Optimality: optimal decisions in the initial period induce a future state, from which (future) decisions are optimal (Bellman, 1957)

$\equiv V^1(a_0,r_1;.)$, value function starting from period 1.

The value resulting from optimal decisions starting from period 1.

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Recursive representation of consumer problem

$$V^0(a_{-1}, r_0; .) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c_0 - a_0 \right) + \beta \cdot V^1(a_0, r_1; .) \right\}$$

- **Bellman Equation**
- **Can analyze optimization problem for period zero…**
  - **…given Bellman Principle of Optimality holds**
  - **(But how do $V^0(.)$ and $V^1(.)$ relate to each other?)**

BELLMAN EQUATION

- **Bellman Equation**

$$V^0(a_{-1}, r_0; .) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c_0 - a_0 \right) + \beta \cdot V^1(a_0, r_1; .) \right\}$$

  - **Starting point for recursive analysis**
  - **Applicable to finite *T*-period or *T* → ∞ problems**
  - **Construction requires identifying state variables of optimization problem**

Deterministic Dynamic Programming

BELLMAN EQUATION

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  - **Starting point for recursive analysis**
  - **Applicable to finite *T*-period or *T* → ∞ problems**
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- ***T*-period problem**
  - **Solution involves sequence of functions $V^0(.)$, $V^1(.)$, … , $V^{T-1}(.)$, $V^T(.)$**
  - **$V^i(.)$ functions in general will differ – reflecting time until end of planning horizon**
  - **E.g., maximized value starting from age = 60 different from maximized value starting from age = 30 (intuitively)**

- **Infinite-horizon problem**
  - **Deterministic case: $V(.) \equiv V^i(.) = V^j(.)\ \forall\ i,j$**
  - **Always an infinity of periods left to go**

Stochastic case?

Requires more structure…

BELLMAN EQUATION

- **Bellman Equation (for $T \rightarrow \infty$)**

$$V(a_{-1}, r_0;.) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c_0 - a_0 \right) + \beta \cdot V(a_0, r_1;.) \right\}$$

- **Use to characterize optimal decisions**
- **Period-0 FOCs**

c_0: $u'(c_0) - \lambda_0 = 0$

a_0: $-\lambda_0 + \beta V_1(a_0, r_1;.) = 0$ **How to compute $V_1(.)$?**

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Return to this …

- ***Suppose*** **optimal choice characterized by $c_0 = c(a_{-1};.)$, $a_0 = a(a_{-1};.)$ ($c(.)$ and $a(.)$ time-invariant functions in infinite-period problem)**
  - **Insert in value function (can now drop max operator)**

$$V(a_{-1}, r_0;.) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1;.)$$

  - **Now compute marginal**

Bellman Equation

- **Bellman Equation (for *T* → ∞)**

$$V(a_{-1}, r_0;.) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1;.)$$

  - **Now compute marginal (suppress *r* argument of *c*(.) and *a*(.) functions)**

$$V_1(a_{-1}, r_0;.) =$$

Bellman Equation

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$$V_1(a_{-1}, r_0;.) =$$

$$\Rightarrow \quad V_1(a_{-1}, r_0;.) = \lambda_0 (1+r_{-1})$$

Envelope Condition

- **Envelope Theorem**
  - **In computing first-order effects of changes in a problem's parameters on the maximized value, can ignore how optimal choices will adjust**
    - **Intuition: because already at a max (marginal costs = marginal benefits)**
  - **Need only consider the direct effect**

Note: envelope theorem has nothing to do with dynamic programming

Bellman Equation

- **Bellman Equation (for $T \to \infty$)**

$$V(a_{-1}, r_0;.) \equiv u(c(a_{-1})) + \lambda_0\left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1})\right) + \beta \cdot V(a(a_{-1}), r_1;.)$$

  - **Now compute marginal (suppress *r* argument of *c*(.) and *a*(.) functions)**

$$V_1(a_{-1}, r_0;.) =$$

$$\Rightarrow \quad V_1(a_{-1}, r_0;.) = \lambda_0(1+r_{-1}) \xrightarrow{\text{evaluate at period 1}} V_1(a_0, r_1;.) = \lambda_1(1+r_0)$$ **Envelope Condition**

- **Envelope Theorem**
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- **Use to characterize optimal decisions**

- **Period-0 FOCs, now evaluated using $c(a_{-1})$, $a(a_{-1})$**

  c_0: $u'(c(a_{-1})) - \lambda_0 = 0$

  a_0: $-\lambda_0 + \beta V_1(a_0(a_{-1}), r_1;.) = 0$

  Env: $V_1(a(a_{-1}), r_1;.) = \lambda_1(1+r_0)$

Bellman Equation

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Env: $V_1(a(a_{-1}), r_1; .) = \lambda_1 (1 + r_0)$

$$\Rightarrow u'(c(a_{-1})) = \beta(1+r_0) u'(c(a_0))$$

- **Seems like usual Euler equation from sequential analysis (deterministic)…**

Deterministic – Recursive Analysis

- **Solution of infinite-horizon consumer problem (starting from date zero)…**
- **…is a consumption decision rule $c(a_{-1}; .)$, asset decision rule $a(a_{-1}; .)$, and value function $V(a_{-1}; .)$ that satisfies**
  - **Bellman equation**

$$V(a_{-1}, r_0; .) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1; .)$$

  - **Euler equation**

$$u'(c(a_{-1})) = \beta V_1(a(a_{-1}), r_1; .) \quad \xleftrightarrow{\text{by envelope theorem}} \quad u'(c(a_{-1})) = \beta(1+r_0) u'(c(a_{-1}))$$

    - **which is the TVC in the limit $t \rightarrow \infty$:** $\lim_{t \to \infty} \beta^t u'(c(a^*_{t-1})) \cdot a(a^*_{t-1}) = 0$

  - **Budget constraint**

$$y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) = 0$$

taking as given (a_{-1}, r_0, r_{-1})

Deterministic – Sequential Analysis

- Solution of infinite-horizon consumer problem (starting from date zero)…
- is a consumption and asset sequence $\{c_t^*, a_t^*\}_{t=0}^{\infty}$ that satisfies
  - Sequence of flow budget constraints

$$c_t^* + a_t^* = y_t + (1+r_{t-1})a_{t-1}^*, \quad t = 0,1,2,\ldots$$

  - Sequence of Euler equations

$$u'(c_t^*) = \beta u'(c_{t+1}^*)(1+r_t), \quad t = 0,1,2,\ldots$$

    - which is the TVC in the limit $t \to \infty$: $\lim_{t\to\infty} \beta^t u'(c_t^*)a_t^* = 0$

taking as given $\left(\{r_t, y_t\}_{t=0}^{\infty}, a_{-1}, r_{-1}\right)$

Does solution to recursive problem coincide with solution to sequential problem?

Recursive vs. Sequential Analysis

- **Does solution to recursive problem coincide with solution to sequential problem?**
- **Does solution to sequential problem coincide with solution to recursive problem?**
- ?…

RECURSIVE VS. SEQUENTIAL ANALYSIS

- Does solution to recursive problem coincide with solution to sequential problem?
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- ?...
- In constructing Bellman representation ($T \rightarrow \infty$ case), the imposition of time-invariant functions $c(a)$, $a(a)$ potentially limits the class of solutions
  - In original sequential formulation, this is neither explicitly nor implicitly a requirement of the solution!
- In general (here without proof…)
  - Solution to the sequential problem is also a solution to the recursive problem
  - Solution to the recursive problem is also a solution to the sequential problem provided some further regularity conditions hold
- Stokey, Lucas, Prescott text (1989)

RECURSIVE VS. SEQUENTIAL ANALYSIS

- So why go recursive?
  - Allows application of series of theorems/results that guarantee a solution exists in the space of functions
  - Allows application of series of theorems/results that help find solution in the space of functions

Underlying theory:
Contraction Mapping Theorem, Blackwell's Sufficient Conditions for a Contraction, Theorem of the Maximum

THEORY

- **Blackwell's Sufficient Conditions for a Contraction:** Let X be a subset of R^l and let $B(X)$ be the set of bounded functions $f : X \rightarrow R$ with the sup norm. Let $T : B(X) \rightarrow B(X)$ be an operator satisfying

  a. (Monotonicity) $f, g \in B(X)$ and $f(x) \leq g(x)$, for all $x \in X$, implies $(Tf)(x) \leq (Tg)(x)$, for all $x \in X$
  b. (Discounting) There exists some $\beta \in (0,1)$ such that

  $[T(f+a)](x) \leq (Tf)(x) + \beta a$, for all $f \in B(X)$, $a \geq 0$, $x \in X$

  Then T is a contraction with modulus β.

  (Note: $(f+a)(x)$ is the function defined by $(f+a)(x) = f(x) + a$)

THEORY

- Let (S, ρ) be a metric space and $T : S \rightarrow S$ be a function mapping set S into itself. T is a **contraction mapping (with modulus β)** if for some $\beta \in (0,1)$, $\rho(Tx, Ty) \leq \beta\rho(x, y)$ for all $x, y \in S$.
  Example: $S = [a, b]$ with $\rho(x, y) = |x - y|$ (Euclidean norm)

- **Contraction Mapping Theorem:** If (S, ρ) is a metric space and $T : S \rightarrow S$ is a contraction mapping with modulus β, then
  a. T has exactly one fixed point v in set S.
  b. For any $v_0 \in S$, $\rho(T^n v_0, v) \leq \beta\rho(x, y)$ for $n = 0, 1, 2, \ldots$

- CMT states that a contraction mapping has a **unique fixed point,** and the fixed point can be found by iterative application of the mapping T starting starting from any point in S.

THEORY

- **General class of problems to which our (usual) economic optimization problems belong have the form**

  $(Tv)(x) = \sup_{y \in \Gamma(x)} [F(x,y) + \beta v(y)]$

- **For our economic theory: would like operator *T* to map the space *C*(*X*) of bounded continuous functions of the state vector into itself. Would also like to be able to characterize the set of maximizing values of *y* given *x*.**

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- **Theorem of the Maximum: Let *X* be a subset of R^l, *Y* be a subset of R^m, let $f : X \times Y \rightarrow R$ be a (single-valued) continuous function, and let $\Gamma : X \rightarrow Y$ be a compact-valued and continuous correspondence. The problem we are interested in is of the form $\sup_{y \in \Gamma(x)} f(x,y)$. Then**

  - ***a.*** ***sup*** **can be replaced with *max* because, for each *x*, the maximum is attained and the function $h(x) = \max_{y \in \Gamma(x)} f(x,y)$ is well defined and continuous**
  - **b.** **The correspondence $G(x) = y \in \Gamma(x) : f(x,y) = h(x)$ is well defined, is non-empty, is compact-valued, and upper hemi-continuous.**

- **Theorem of the Maximum establishes the existence of the maximum of the problem.**

THEORY

- Suppose in addition to the hypotheses of the Theorem of the Maximum, the correspondence Γ is convex-valued and the function f is strictly concave in y.

  → Then G is single-valued. Call this function g, and g is continuous.

- Establishes that, given these conditions and given the unique solution of the Bellman Equation, there is a unique g that is the optimal "decision rule."

- If $\{f_n(x,y)\}$ is a sequence of continuous functions converging to $f(x,y)$, each strictly concave in y, then the sequence of functions $\{g_n(x)\}$ (which are the argmax of the sequence $\{f_n(x,y)\}$) converges pointwise to $g(x)$, which is the argmax of $f(x,y)$.

- The latter result is very useful considered in the context of the Contraction Mapping Theorem. It guarantees that the solutions to the sequence of problems converges to the true solution.

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Underlying theory:
Contraction Mapping Theorem, Blackwell's Sufficient Conditions for a Contraction, Theorem of the Maximum

  - Computational algorithms require it – computers can't handle infinite-dimensional objects!
    - Soon: simple computational algorithms

- Can't really "choose" whether want to analyze problem sequentially or recursively
  - All but the most limited of problems/models require computational solution
  - In which case model analysis is recursive

- What about stochastic dynamic programming?
  - Even more structure required....

STOCHASTIC DYNAMIC PROGRAMMING

- **Even more structure required on the problem to recursively solve dynamic stochastic optimization problems**

- **Main (new) technical problem**
  - **Branching of event tree at each of *T* periods (possibly $T \rightarrow \infty$)**

- **Main technical solution/assumption**
  - **Assume risk follows Markov process**
  - **Which enables series of theorems/results from deterministic dynamic programming to work in stochastic case…**
  - **…given further technical regularity assumptions**