LINEAR APPROXIMATION OF THE BASELINE RBC MODEL

FEBRUARY 1, 2012

LINEARIZATION OF THE RBC MODEL

- **For** $f(x,y,z)=0$**, multivariable Taylor linear expansion around** $(\bar{x},\bar{y},\bar{z})$

$$f(x,y,z)\approx f(\bar{x},\bar{y},\bar{z})+f_x(\bar{x},\bar{y},\bar{z})(x-\bar{x})+f_y(\bar{x},\bar{y},\bar{z})(y-\bar{y})+f_z(\bar{x},\bar{y},\bar{z})(z-\bar{z})$$

- **Four equations describe the dynamic solution to RBC model**
  - **Consumption-leisure efficiency condition**

$$-\frac{u_n(c_t,n_t)}{u_c(c_t,n_t)}=z_t m_n(k_t,n_t)$$

  - **Consumption-investment efficiency condition**

$$u_c(c_t,n_t)=\beta E_t\left[u_c(c_{t+1},n_{t+1})\left(1-\delta+z_{t+1}m_k(k_{t+1},n_{t+1})\right)\right]$$

  - **Aggregate resource constraint**

$$c_t+k_{t+1}-(1-\delta)k_t=z_t m(k_t,n_t)$$

  - **Law of motion for TFP**

$$\ln z_{t+1}=(1-\rho_z)\ln\bar{z}+\rho_z\ln z_t+\varepsilon_{t+1}^{z}$$

STEADY STATE

- **Deterministic steady state the natural local point of approximation**
- **Shut down all shocks and set exogenous variables at their means**
- **The Idea: Let economy run for many (infinite) periods**
  - **Time eventually "doesn't matter" any more**
  - **Drop all time indices**

$$-\frac{u_n(\overline{c},\overline{n})}{u_c(\overline{c},\overline{n})}=\overline{z}m_n(\overline{k},\overline{n})$$

$$u_c(\overline{c},\overline{n})=\beta u_c(\overline{c},\overline{n})\left[m_k(\overline{k},\overline{n})+1-\delta\right]$$

$$\overline{c}+\delta\overline{k}=\overline{z}m(\overline{k},\overline{n})$$

$$\ln\overline{z}=(1-\rho_z)\ln\overline{z}+\rho_z\ln\overline{z}\Rightarrow\overline{z}=\overline{z}\quad\text{(a parameter of the model)}$$

- **Given functional forms and parameter values, solve for (*c*, *n*, *k*)**
  - **The steady state of the model**
  - **Taylor expansion around this point**

LINEARIZATION ALGORITHMS

- **Schmitt-Grohe and Uribe (2004 *JEDC*)**
  - **A perturbation algorithm**
    - **A class of methods used to find an approximate solution to a problem that cannot be solved exactly, by starting from the exact solution of a related problem**
    - **Applicable if the problem can be formulated by adding a "small" term to the description of the exactly-solvable problem**
  - **Matlab code available through Columbia Dept. of Economics web site**

- **Uhlig (1999, chapter in *Computational Methods for the Study of Dynamic Economies*)**
  - **Uses a generalized eigen-decomposition**
    - **Typically implemented with Schur decomposition (Sims algorithm)**
  - **Matlab code available at**

    http://www2.wiwi.hu-berlin.de/institute/wpol/html/toolkit.htm

LINEARIZATION OF THE RBC MODEL

Define co-state vector and state vector

$$y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix} \qquad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$

LINEARIZATION OF THE RBC MODEL

Define co-state vector and state vector

$$y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix} \qquad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$

Order model’s dynamic equations in a vector $\equiv f(y_{t+1}, y_t, x_{t+1}, x_t) = 0$

Consumption-leisure efficiency condition

Consumption-investment efficiency condition

Aggregate resource constraint

Law of motion for TFP

LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

1. **Differentiate** $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** y_{t+1}

First derivatives with respect to:

$$\begin{array}{l} \\ \text{Consumption-leisure efficiency condition} \\ \text{Consumption-investment efficiency condition} \\ \text{Aggregate resource constraint} \\ \text{Law of motion for TFP} \end{array} \begin{array}{c} \begin{array}{cc} c_{t+1} & n_{t+1} \end{array} \\ \left[\begin{array}{cc} & \\ & \\ & \\ & \end{array}\right] \end{array} = f_{y_{t+1}}$$

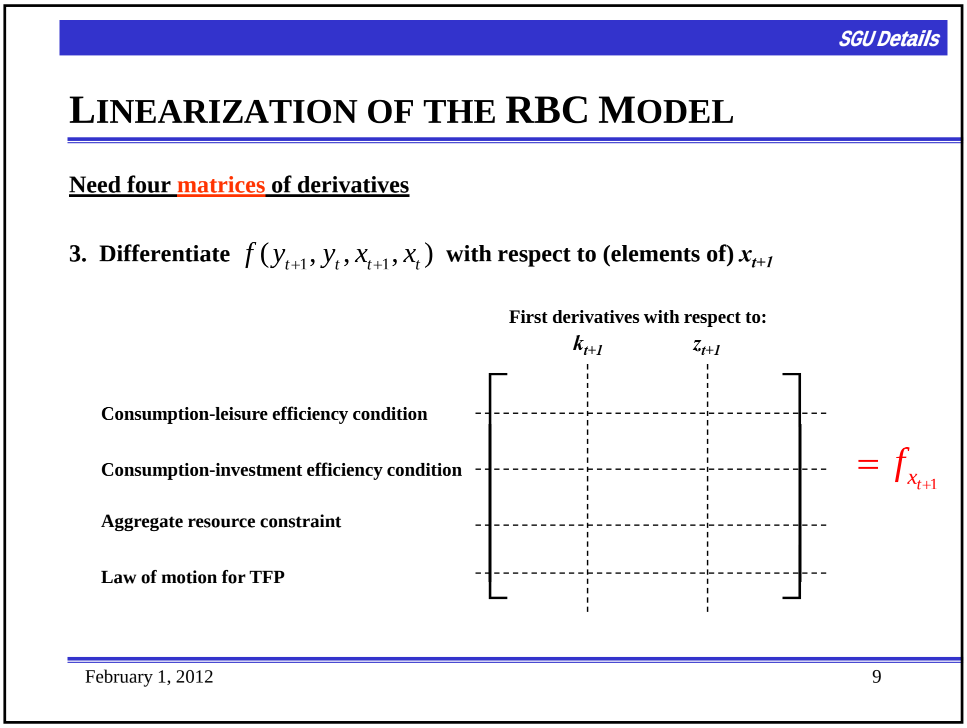
LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

2. **Differentiate** $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** y_t

First derivatives with respect to:

$$\begin{array}{l} \\ \text{Consumption-leisure efficiency condition} \\ \text{Consumption-investment efficiency condition} \\ \text{Aggregate resource constraint} \\ \text{Law of motion for TFP} \end{array} \begin{array}{c} \begin{array}{cc} c_t & n_t \end{array} \\ \left[\begin{array}{cc} & \\ & \\ & \\ & \end{array}\right] \end{array} = f_{y_t}$$



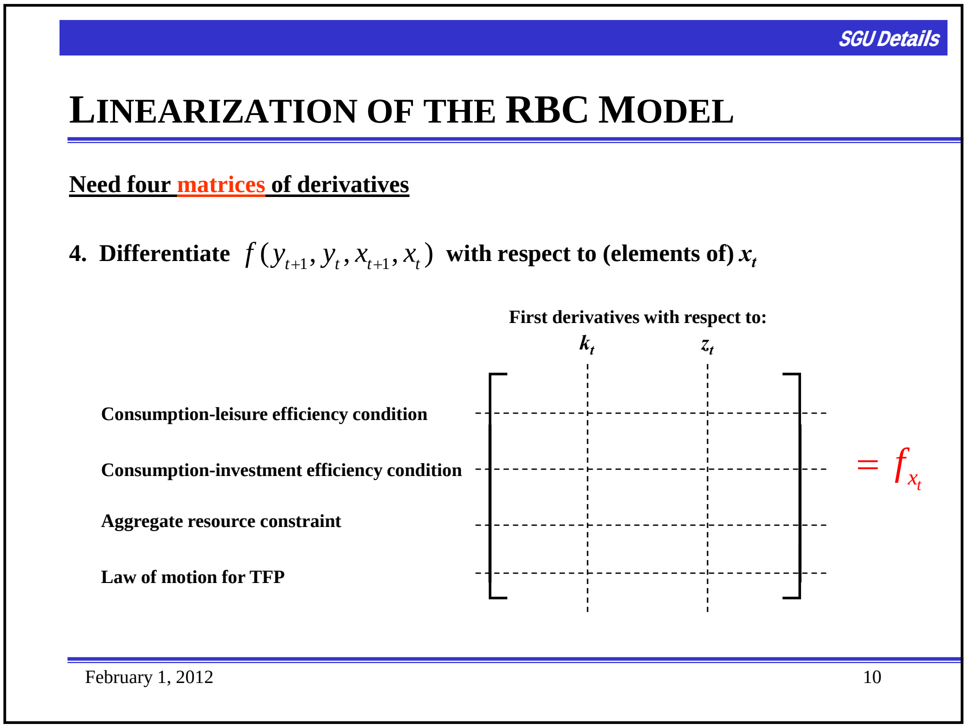
LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

3. Differentiate $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** x_{t+1}

First derivatives with respect to:

$$\begin{array}{l|cc|} & k_{t+1} & z_{t+1} \\ \text{Consumption-leisure efficiency condition} & & \\ \text{Consumption-investment efficiency condition} & & \\ \text{Aggregate resource constraint} & & \\ \text{Law of motion for TFP} & & \end{array} = f_{x_{t+1}}$$



LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

4. Differentiate $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** x_t

First derivatives with respect to:

$$\begin{array}{l|cc|} & k_t & z_t \\ \text{Consumption-leisure efficiency condition} & & \\ \text{Consumption-investment efficiency condition} & & \\ \text{Aggregate resource constraint} & & \\ \text{Law of motion for TFP} & & \end{array} = f_{x_t}$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]=E_t\begin{bmatrix} f^1(y_{t+1},y_t,x_{t+1},x_t) \\ f^2(y_{t+1},y_t,x_{t+1},x_t) \\ f^3(y_{t+1},y_t,x_{t+1},x_t) \\ f^4(y_{t+1},y_t,x_{t+1},x_t) \end{bmatrix}$$

Consumption-leisure efficiency condition

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Consumption-leisure efficiency condition

Consumption-investment efficiency condition

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Law of motion for TFP

Conjecture equilibrium decision rules

Note: g(.) and h(.) are time invariant functions!

"Perturbation parameter": governs size of shocks

$$y_t = g(x_t,\sigma)$$
$$x_{t+1} = h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1}$$

Matrix of standard deviations of state variables

Substitute decision rules into dynamic equations

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]=0$$

$$=E_t\left[f(g(x_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$

$$=E_t\left[f(g(h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$

$$\equiv F(x_t,\sigma)$$

$$\Downarrow$$

$$F_x(x_t,\sigma)=0 \qquad F_\sigma(x_t,\sigma)=0$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

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$$=E_t\left[f(g(h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$

$$\equiv F(x_t,\sigma)$$

$$\Downarrow$$

Using chain rule and suppressing arguments

$$F_x(x_t,\sigma)=$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]=0$$

$$=E_t\left[f(g(x_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$

$$=E_t\left[f(g(h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$

$$\equiv F(x_t,\sigma)$$

$$\Downarrow$$

Using chain rule and suppressing arguments

$$F_x(x_t,\sigma)=f_{y_{t+1}}\cdot g_x\cdot h_x+f_{y_t}\cdot g_x+f_{x_{t+1}}\cdot h_x+f_{x_t}$$

$$=0$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]=0$$

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$$\equiv F(x_t,\sigma)$$

$$\Downarrow$$

Using chain rule and suppressing arguments

$$F_x(x_t,\sigma)=f_{y_{t+1}}\cdot g_x\cdot h_x+f_{y_t}\cdot g_x+f_{x_{t+1}}\cdot h_x+f_{x_t}$$

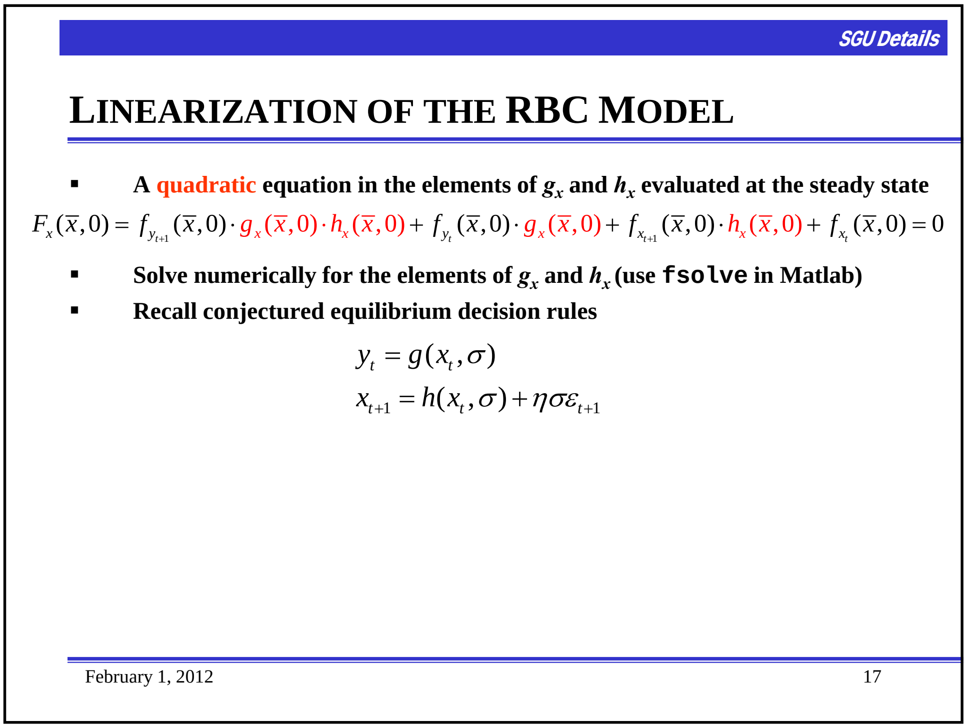
$$=0$$

Setting $\sigma = 0$ shuts down shocks

Holds, in particular, at the deterministic steady state $(\bar{x},0)$

$$F_x(\bar{x},0)=f_{y_{t+1}}\cdot g_x\cdot h_x+f_{y_t}\cdot g_x+f_{x_{t+1}}\cdot h_x+f_x=0$$

Each term is evaluated at the steady state – just as Taylor theorem requires



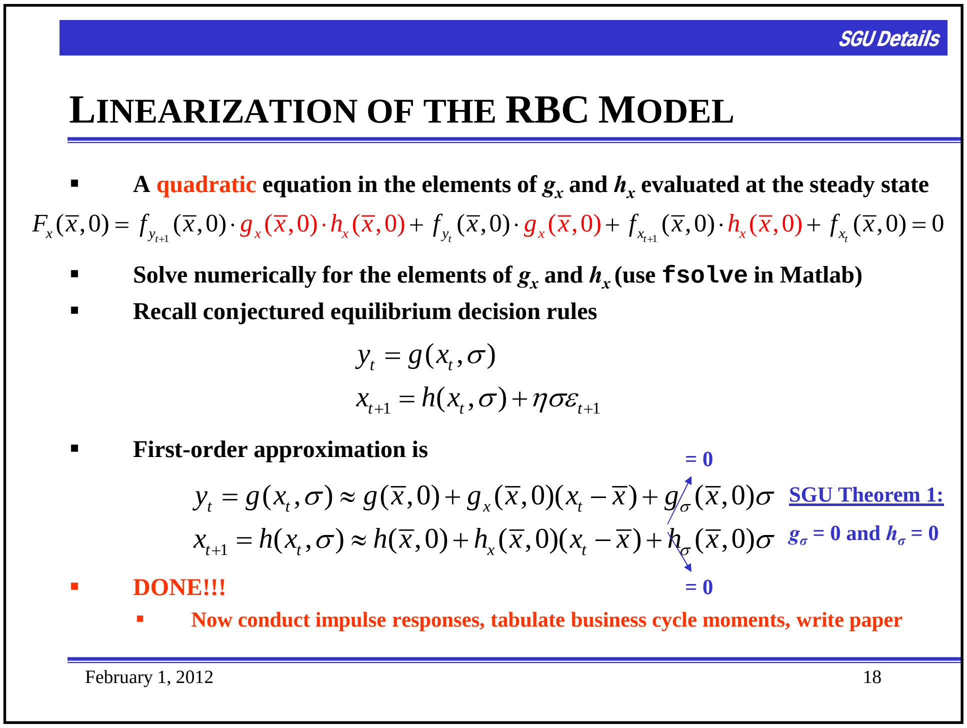
LINEARIZATION OF THE RBC MODEL

- **A** **quadratic** **equation in the elements of g_x and h_x evaluated at the steady state**

$$F_x(\bar{x},0) = f_{y_{t+1}}(\bar{x},0)\cdot g_x(\bar{x},0)\cdot h_x(\bar{x},0) + f_{y_t}(\bar{x},0)\cdot g_x(\bar{x},0) + f_{x_{t+1}}(\bar{x},0)\cdot h_x(\bar{x},0) + f_{x_t}(\bar{x},0) = 0$$

- **Solve numerically for the elements of g_x and h_x (use `fsolve` in Matlab)**
- **Recall conjectured equilibrium decision rules**

$$y_t = g(x_t,\sigma)$$
$$x_{t+1} = h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}$$



SGU Details

LINEARIZATION OF THE RBC MODEL

- **A** **quadratic** **equation in the elements of g_x and h_x evaluated at the steady state**

$$F_x(\bar{x},0) = f_{y_{t+1}}(\bar{x},0)\cdot g_x(\bar{x},0)\cdot h_x(\bar{x},0) + f_{y_t}(\bar{x},0)\cdot g_x(\bar{x},0) + f_{x_{t+1}}(\bar{x},0)\cdot h_x(\bar{x},0) + f_{x_t}(\bar{x},0) = 0$$

- **Solve numerically for the elements of g_x and h_x (use `fsolve` in Matlab)**
- **Recall conjectured equilibrium decision rules**

$$y_t = g(x_t,\sigma)$$
$$x_{t+1} = h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}$$

- **First-order approximation is**

$$y_t = g(x_t,\sigma) \approx g(\bar{x},0) + g_x(\bar{x},0)(x_t - \bar{x}) + \overset{= 0}{\cancel{g_\sigma}}(\bar{x},0)\sigma$$
$$x_{t+1} = h(x_t,\sigma) \approx h(\bar{x},0) + h_x(\bar{x},0)(x_t - \bar{x}) + \underset{= 0}{\cancel{h_\sigma}}(\bar{x},0)\sigma$$

SGU Theorem 1:
$g_\sigma = 0$ and $h_\sigma = 0$

- **DONE!!!**
  - **Now conduct impulse responses, tabulate business cycle moments, write paper**

CERTAINTY EQUIVALENCE

- **Displayed by a model if decision rules do not depend on the standard deviation of exogenous uncertainty – e.g., PRECAUTIONARY SAVINGS!**
- **For stochastic problems with quadratic objective function and linear constraints, the decision rules are identical to those of the nonstochastic problem**
- **Here, we have**

$$y_t = g(x_t,\sigma) \approx g(\bar{x},0) + g_x(\bar{x},0)(x_t-\bar{x}) + \overset{=0}{g_\sigma}(\bar{x},0)\sigma$$

$$x_{t+1} = h(x_t,\sigma) \approx h(\bar{x},0) + h_x(\bar{x},0)(x_t-\bar{x}) + \underset{=0}{h_\sigma}(\bar{x},0)\sigma$$

- **SGU Theorem 1: $g_\sigma = 0$ and $h_\sigma = 0$**
    - **First-order approximated decision rules do not depend on the size of the shocks, which is governed by σ**
    - **Not the same thing as "exact CE," but refer to it as CE**

LINEARIZING THE RBC MODEL

- **Assume** $u(c_t,n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t,n_t) = k_t^{\alpha} n_t^{1-\alpha}$
- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$
- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \frac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $\quad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ **)**

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- **Compute first row of matrix** f_{yt+1}

$$\begin{array}{l c} & \begin{array}{cc} c_{t+1} & n_{t+1} \end{array} \\ \begin{array}{l} \textbf{Consumption-leisure efficiency condition} \\ \textbf{Consumption-investment efficiency condition} \\ \textbf{Aggregate resource constraint} \\ \textbf{Law of motion for TFP} \end{array} & \left[\begin{array}{c:c} & \\ \hdashline & \\ \hdashline & \\ \hdashline & \end{array}\right] \end{array}$$

LINEARIZING THE RBC MODEL

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- **Compute first row of matrix** f_{yt}

$$\begin{array}{l c}
 & \begin{array}{cc} \boldsymbol{c_t} & \boldsymbol{n_t} \end{array} \\
\begin{array}{l}
\textbf{Consumption-leisure efficiency condition} \\
\textbf{Consumption-investment efficiency condition} \\
\textbf{Aggregate resource constraint} \\
\textbf{Law of motion for TFP}
\end{array} &
\begin{bmatrix}
\dfrac{\psi}{n_t} & -\dfrac{\psi c_t}{n_t^2} + \alpha(1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha-1} \\
 & \\
 & \\
 &
\end{bmatrix}
\end{array}$$

LINEARIZING THE RBC MODEL

- **Assume** $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
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- **Compute first row of matrix** f_{xt+1}

$$\begin{array}{l c}
 & \begin{array}{cc} \boldsymbol{k_{t+1}} & \boldsymbol{z_{t+1}} \end{array} \\
\begin{array}{l}
\textbf{Consumption-leisure efficiency condition} \\
\textbf{Consumption-investment efficiency condition} \\
\textbf{Aggregate resource constraint} \\
\textbf{Law of motion for TFP}
\end{array} &
\begin{bmatrix}
0 & 0 \\
 & \\
 & \\
 &
\end{bmatrix}
\end{array}$$

(Partial) Example

LINEARIZING THE RBC MODEL

- **Assume** $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
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- **Compute first row of matrix** f_{xt}

$$
\begin{array}{lc}
 & \begin{array}{cc} \boldsymbol{k_t} & \boldsymbol{z_t} \end{array} \\
\begin{array}{l}
\textbf{Consumption-leisure efficiency condition} \\
\textbf{Consumption-investment efficiency condition} \\
\textbf{Aggregate resource constraint} \\
\textbf{Law of motion for TFP}
\end{array}
&
\left[\begin{array}{c:c}
-\alpha(1-\alpha) z_t \dfrac{k_t^{\alpha-1}}{n_t^{\alpha}} & -(1-\alpha) \dfrac{k_t^{\alpha}}{n_t^{\alpha+1}} \\ \hdashline
 & \\ \hdashline
 & \\ \hdashline
 &
\end{array}\right]
\end{array}
$$

(Partial) Example

LINEARIZING THE RBC MODEL

- **In deterministic steady state, the first rows of** $f_{yt+1}, f_{yt}, f_{xt+1}, f_{xt}$ **are**

f_{yt+1}	0	0
f_{yt}	$\dfrac{\psi}{\overline{n}}$	$-\dfrac{\psi \overline{c}}{\overline{n}^2} + \alpha(1-\alpha)\overline{z}\overline{k}^{\alpha}\overline{n}^{-\alpha-1}$
f_{xt+1}	0	0
f_{xt}	$-\alpha(1-\alpha)\overline{z}\dfrac{\overline{k}^{\alpha-1}}{\overline{n}^{\alpha}}$	$-(1-\alpha)\dfrac{\overline{k}^{\alpha}}{\overline{n}^{\alpha+1}}$

- **How to compute derivatives** f_{yt+1}, f_{yt}, f_{xt+1}, f_{xt}**?**
  - **By hand (feasible for small models)**
  - **Schmitt-Grohe and Uribe Matlab analytical routines**
  - **Your own Maple or Mathematica programs**
  - **Dynare package**

CALIBRATION?

- **Solving for the steady state?**
- **Choosing parameter values?**
- **Next: calibration of the baseline representative-agent (RBC + growth) model**