Balanced Growth Path in Labor Search and Matching Model

January 9, 2017

CALIBRATION – PHILOSOPHY

- An economic model is a measuring device
- If model makes "believable" predictions along some important dimensions (i.e., "matches some key data")…
- …then maybe its predictions are "believable" along the novel dimensions of the model
- Getting some "partial derivatives" of the model in known directions correct…
- …may build credibility that its "partial derivatives" in novel directions are at least not grossly incorrect
- Make model match some data of interest – often long-run (i.e., time-averaged data) growth facts
  - Preferably well-accepted "stylized facts"
  - Solow growth model in the background
  - Natural candidate: Kaldor growth facts
- Calibration vs. Estimation

Calibration of Baseline RBC Model

- Must take a stand on three (related) points
  - Which data do we want model to match? (even constructing data is challenging…)
  - Functional forms (utility, production)
  - Parameter values

- Choose functional forms consistent with **"Kaldor-plus facts"**
  - **(K1)** Capital income share and labor income share of GDP are stationary
  - **(K2)** All real quantity variables grow at same rate in the long run **("great ratios")**
  - **(K3)** Real interest rate is stationary
  - **(K4)** Hours per worker are stationary
  - **(K5)** (K2) requires trend productivity to be labor-augmenting (Phelps 1966)

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- Often start with RBC model that abstracts from long-run growth
- **But "true" calibration begins with model featuring *only* long-run growth**
  - Puts restrictions on instantaneous utility and production forms
  - Use **(K1)-(K5)** to obtain these restrictions

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- Often start with RBC model that abstracts from long-run growth
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- Richer models: more calibration targets and/or treating data differently
  - Monopoly markups (e.g., Dixit-Stiglitz and sticky price models)
  - Probability of finding a job (e.g., labor matching models)
  - Durable consumption vs. non-durable consumption

SEARCH MODEL WITH GROWTH

- **Absent shocks, TFP grows at deterministic rate *γ***
- **Planner problem / perfect competition**

$$\max E_0 \sum_{t=0}^{\infty} b^t u(C_t, lfp_t) \quad \textbf{subject to}$$

Trend productivity is labor-augmenting (Harrod-neutral) (Makes use of fact (K5))

$$C_t + K_{t+1} - (1-\delta)K_t + \Omega_t \cdot v_t = z_t F(K_t, n_t X_t)$$

Flow resource constraint

Red indicates variables or parameters that will be modified when detrending the model

$$n_t = (1-\rho)n_{t-1} + m(lfp_t - (1-\rho)n_{t-1}, v_t)$$

Aggregate LOM of labor

$$X_{t+1} = \gamma_t X_t, \qquad \gamma \geq 1$$

Evolution of deterministic component of productivity

given stochastic process for evolution of z_t and (K_{-1}, n_{-1}, z_0, X_0)

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Red indicates variables or parameters that will be modified when detrending the model

- **Suppose z_t = 1 always, so only deterministic growth**
- **Deterministic dynamics of (C_t, K_{t+1}, lfp_t, n_t, v_t, X_t) governed by**

(1) $$-\frac{u_{lfp}(C_t, lfp_t)}{u_C(C_t, lfp_t)} = \Omega_t \cdot \frac{m_s(s_t, v_t)}{m_v(s_t, v_t)}$$ **Labor supply function**

$$\frac{\Omega_t}{m_v(s_t, v_t)} = X_t F_2(K_t, n_t X_t)$$ **Labor demand function**

(2) $$\frac{u_C(C_t, lfp_t)}{b u_C(C_{t+1}, lfp_{t+1})} = F_1(K_{t+1}, n_{t+1} X_{t+1}) + 1 - \delta$$ **Capital supply function**

(3) $$+(1-\rho)\left(\frac{b u_c(C_{t+1}, lfp_{t+1})}{u_c(C_t, lfp_t)}\right) \cdot \frac{\Omega_{t+1}\left(1 - m_s(s_{t+1}, v_{t+1})\right)}{m_v(s_{t+1}, v_{t+1})}$$

(4) $$C_t + K_{t+1} - (1-\delta)K_t + \Omega_t \cdot v_t = F(K_t, n_t X_t)$$

(6) $$X_{t+1} = \gamma_t X_t$$ **Normalize X_0 = 1**

(5) $$n_t = (1-\rho)n_{t-1} + m(s_t, v_t)$$ **LOM of labor**

RESTRICTIONS ON FUNCTIONAL FORMS

- **Deterministic dynamics of (C_t, K_{t+1}, lfp_t, n_t, v_t, X_t) governed by**

(1) $$-\frac{u_{lfp}(C_t, lfp_t)}{u_C(C_t, lfp_t)} = \Omega_t \cdot \frac{m_s(s_t, v_t)}{m_v(s_t, v_t)}$$

(2) $$\frac{u_C(C_t, lfp_t)}{bu_C(C_{t+1}, lfp_{t+1})} = F_1(K_{t+1}, n_{t+1}X_{t+1}) + 1 - \delta$$

(3) $$\frac{\Omega_t}{m_v(s_t, v_t)} = X_t F_2(K_t, n_t X_t) + (1-\rho)\left(\frac{bu_C(C_{t+1}, lfp_{t+1})}{u_C(C_t, lfp_t)}\right) \cdot \frac{\Omega_{t+1}\left(1 - m_s(s_{t+1}, v_{t+1})\right)}{m_v(s_{t+1}, v_{t+1})}$$

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- **(K1) Capital income share and labor income share of GDP are stationary**
  And viewing economic profits as zero

$$\Rightarrow F(K, nX) = K^{\alpha}(nX)^{1-\alpha} \qquad (\alpha \approx 0.4)$$

RESTRICTIONS ON FUNCTIONAL FORMS

- **Deterministic dynamics of (C_t, K_{t+1}, lfp_t, n_t, v_t, X_t) governed by**

(1) $$-\frac{u_{lfp}(C_t,lfp_t)}{u_C(C_t,lfp_t)}=\Omega_t\cdot\frac{m_s(s_t,v_t)}{m_v(s_t,v_t)}$$

(2) $$\frac{u_C(C_t,lfp_t)}{bu_C(C_{t+1},lfp_{t+1})}=\alpha\left(\frac{K_{t+1}/X_{t+1}}{n_{t+1}}\right)^{\alpha-1}+1-\delta$$

(3) $$\frac{\Omega_t}{m_v(s_t,v_t)}=X_tF_2(K_t,n_tX_t)+(1-\rho)\left(\frac{bu_C(C_{t+1},lfp_{t+1})}{u_C(C_t,lfp_t)}\right)\cdot\frac{\Omega_{t+1}\left(1-m_s(s_{t+1},v_{t+1})\right)}{m_v(s_{t+1},v_{t+1})}$$

(4) $$C_t+K_{t+1}-(1-\delta)K_t+\Omega_t\cdot v_t=K_t^{\alpha}\left(n_tX_t\right)^{1-\alpha}$$

(5) $$n_t=(1-\rho)n_{t-1}+m(s_t,v_t)$$

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(2) $$\frac{u_C(C_t, lfp_t)}{bu_C(C_{t+1}, lfp_{t+1})} = \alpha\left(\frac{K_{t+1}/X_{t+1}}{n_{t+1}}\right)^{\alpha-1} + 1 - \delta$$

(3) $$\frac{\Omega_t}{m_v(s_t, v_t)} = X_t F_2(K_t, n_t X_t) + (1-\rho)\left(\frac{bu_C(C_{t+1}, lfp_{t+1})}{u_C(C_t, lfp_t)}\right) \cdot \frac{\Omega_{t+1}(1 - m_s(s_{t+1}, v_{t+1}))}{m_v(s_{t+1}, v_{t+1})}$$

(4) $$C_t + K_{t+1} - (1-\delta)K_t + \Omega_t \cdot v_t = K_t^{\alpha}(n_t X_t)^{1-\alpha}$$

(5) $$n_t = (1-\rho)n_{t-1} + m(s_t, v_t)$$

(6) $$X_{t+1} = \gamma_t X_t$$

- **(K2) All real (goods-denominated…) quantity variables grow at same rate in the *extremely* long run**

$$\Rightarrow \frac{Y_{t+1}}{Y_t} = \frac{C_{t+1}}{C_t} = \frac{K_{t+1}}{K_t} = \frac{\Omega_{t+1}}{\Omega_t} = \frac{X_{t+1}}{X_t} = \gamma, \quad \forall t$$

$$\Rightarrow y_t \equiv \frac{Y_t}{X_t} = \bar{y}, \quad k_t \equiv \frac{K_t}{X_t} = \bar{k}, \quad c_t \equiv \frac{C_t}{X_t} = \bar{c}, \;\; \omega_t \equiv \frac{\Omega_t}{X_t} = \bar{\omega} \quad \forall t$$

RESTRICTIONS ON FUNCTIONAL FORMS

- **Deterministic dynamics of (C_t, K_{t+1}, lfp_t, n_t, v_t, X_t) governed by**

(1) $$-\frac{u_{lfp}(C_t, lfp_t)}{u_C(C_t, lfp_t)} = \Omega_t \cdot \frac{m_s(s_t, v_t)}{m_v(s_t, v_t)}$$

(2) $$\frac{u_C(C_t, lfp_t)}{bu_C(C_{t+1}, lfp_{t+1})} = \alpha\left(\frac{K_{t+1}/X_{t+1}}{n_{t+1}}\right)^{\alpha-1} + 1 - \delta$$

(3) $$\frac{\Omega_t}{m_v(s_t, v_t)} = (1-\alpha)X_t\left(\frac{K_t/X_t}{n_t}\right)^{\alpha} + (1-\rho)\left(\frac{bu_C(C_{t+1}, lfp_{t+1})}{u_C(C_t, lfp_t)}\right)\cdot\frac{\Omega_{t+1}(1-m_s(s_{t+1}, v_{t+1}))}{m_v(s_{t+1}, v_{t+1})}$$

Recall *F*(.) is CRS

(4) $$C_t + K_{t+1} - (1-\delta)K_t + \Omega_t \cdot v_t = K_t^{\alpha}\left(n_t X_t\right)^{1-\alpha}$$

(5) $$n_t = (1-\rho)n_{t-1} + m(s_t, v_t)$$

(6) $$X_{t+1} = \gamma_t X_t$$

- **Scale (1), (3), and (4) by X_t to write in stationary form**

(1) $$-\frac{u_{lfp}(C_t, lfp_t)}{u_C(C_t, lfp_t)} = \omega_t \cdot \frac{m_s(s_t, v_t)}{m_v(s_t, v_t)}$$

(2) $$\frac{u_C(C_t, lfp_t)}{bu_C(C_{t+1}, lfp_{t+1})} = \alpha\left(\frac{k_{t+1}}{n_{t+1}}\right)^{\alpha-1} + 1 - \delta$$

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(2) $$\frac{u_C(C_t, lfp_t)}{bu_C(C_{t+1}, lfp_{t+1})} = \alpha\left(\frac{\bar{k}}{n_{t+1}}\right)^{\alpha-1} + 1 - \delta$$

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(1) $$-\frac{u_{lfp}(C_t, lfp_t)}{u_C(C_t, lfp_t)} = \omega_t \cdot \frac{m_s(s_t, v_t)}{m_v(s_t, v_t)}$$

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- **(K4 – generalized) Time spent in each activity is stationary**

$$\Rightarrow lfp_t = \overline{lfp}$$
$$\text{and } s_t = \bar{s}$$
$$\text{and } n_t = \bar{n}$$

along deterministic path.
BUT
***lfp* is endogenous…**
***s* is endogenous…**
***n* is endogenous…**

- **Recall definition: $lfp_t = (1-\rho)n_{t-1} + s_t$**

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(1) $$-\frac{u_{lfp}(C_t,\overline{lfp})}{u_C(C_t,\overline{lfp})}=\omega\cdot\frac{m_s(\overline{s},\overline{v})}{m_v(\overline{s},\overline{v})}$$

(2) $$\frac{u_C(C_t,\overline{lfp})}{bu_C(C_{t+1},\overline{lfp})}=\alpha\left(\frac{\overline{k}}{\overline{n}}\right)^{\alpha-1}+1-\delta$$

(3) $$\frac{\omega}{m_v(\overline{s},\overline{v})}=(1-\alpha)\left(\frac{\overline{k}}{\overline{n}}\right)^{\alpha}+\gamma_t\cdot(1-\rho)\left(\frac{bu_C(C_{t+1},\overline{lfp})}{u_C(C_t,\overline{lfp})}\right)\cdot\frac{\omega\cdot\left(1-m_s(\overline{s},\overline{v})\right)}{m_v(\overline{s},\overline{v})}$$

(4) $$\overline{c}+\gamma_t\overline{k}-(1-\delta)\overline{k}+\omega\cdot v_t=\overline{k}^{\alpha}n_t^{1-\alpha}$$

(5) $$\overline{n}=(1-\rho)\overline{n}+m(\overline{s},\overline{v})$$

(6) $$X_{t+1}=\gamma_t X_t$$

- **Implied already (RHS of (2)) is**
  - **(K3) Real interest rate is stationary**

- **Final step – functional form for utility?**

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- **Observations**
  - **Optimal choice of *lfp* must be independent of X_t (from (1))**
    - **Requires offsetting income and substitution effects of wages (long-run productivity) on LFP (aka, labor supply)**
  - **IMRS can only depend on C_{t+1}/C_t (from (2)), which in turn = γ_t**

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- **Two requirements together imply**

$$u(C_t,lfp_t)=\begin{cases}\dfrac{\left[C_t v(lfp_t)\right]^{1-\sigma}-1}{1-\sigma} & \text{if } \sigma>0,\ \sigma\neq 1\\ \ln C_t+v(lfp_t) & \text{if } \sigma=1\end{cases}$$