Economics 8723
Macroeconomic Theory
Problem Set 2 – Suggested Solutions (Sketch)
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Labor Income Taxes, Nash-Bargained Wages, and Proportionally-Bargained Wages.

In an economy that includes proportional labor income taxes, you will compare and contrast the surplus sharing conditions and the wage outcomes that arise from two different wage-determination mechanisms. Labor income taxation is the only form of taxation in the economy; labor income in period t is taxed at the proportional rate $\tau_t^n \ \forall t$.

Presented below are the payoffs (more precisely, the payoffs out of the total surplus) to each of the two parties relevant for bargaining. The payoff expressions are presented in, at face value, two different forms. Due to the underlying market structure and timing of events that give rise to the following payoff expressions (which is outside the scope of the analysis you will conduct here) the two "different" formulations are actually not different at all – the underlying market structure ensures that they are isomorphic.[1]

Thus, for the analysis to be conducted here, take the following payoff expressions as given.

Defining the one-period ahead stochastic discount factor as $\Xi_{t+1|t} \equiv \dfrac{\beta u'(c_{t+1})}{u'(c_t)}$, the period-t payoff expressions relevant for wage determination are

$$\mathbf{W}(w_t) - \mathbf{U}_t = (1-\tau_t^n)w_t - b + (1-\rho_x)E_t\left\{\Xi_{t+1|t}(1-k_{t+1}^h)\left(\mathbf{W}(w_{t+1}) - \mathbf{U}_{t+1}\right)\right\} \qquad (1)$$

and

$$\mathbf{J}(w_t) - \mathbf{V}_t = z_t - w_t + (1-\rho_x)E_t\left\{\Xi_{t+1|t}\left(\mathbf{J}(w_{t+1}) - \mathbf{V}_{t+1}\right)\right\} \qquad (2)$$

for, respectively, the potential new employee and the potential new employer.

[1] In class, we will study the underlying market structure and timing of events that gives rise to the value equations. In the analysis here, take the value equations as given.

Due to the underlying market structure and timing of events, the period-t payoff expressions can be equivalently (i.e., isomorphically) expressed as

$$\underbrace{\frac{h'(lfp_t)-u'(c_t)b}{k_t^h\cdot u'(c_t)}}_{\equiv \mathbf{W}(w_t)-\mathbf{U}_t} = (1-\tau_t^n)w_t - b + (1-\rho_x)E_t\left\{\Xi_{t+1|t}\left[\left(1-k_{t+1}^h\right)\underbrace{\left(\frac{h'(lfp_{t+1})-u'(c_{t+1})b}{k_{t+1}^h\cdot u'(c_{t+1})}\right)}_{\equiv \mathbf{W}(w_{t+1})-\mathbf{U}_{t+1}}\right]\right\} \quad (3)$$

and

$$\underbrace{\frac{\gamma}{k_t^f}-0}_{\equiv \mathbf{J}(w_t)-\mathbf{V}_t} = z_t - w_t + (1-\rho_x)E_t\left\{\Xi_{t+1|t}\underbrace{\left(\frac{\gamma}{k_{t+1}^f}-0\right)}_{\equiv \mathbf{J}(w_{t+1})-\mathbf{V}_{t+1}}\right\}. \quad (4)$$

One conventional view on how wages are determined in the search and matching framework is via **“Nash bargaining.”** Another conventional view on how wages are determined in the search and matching framework is via **“Proportional Bargaining.”** In both of these wage determination systems, the wage is determined **after** new matches have been formed (i.e., ex-post of the matching process).

In Nash bargaining, the two parties (the potential new employee and the potential new employer) **jointly** decide (“negotiate”) the real wage w_t based on maximizing the following expression

$$\text{Nash maximand} = \underbrace{\left(\mathbf{W}(w_t)-\mathbf{U}_t\right)^{\eta}}_{\text{worker's payoff from total surplus}} \underbrace{\left(\mathbf{J}(w_t)-\mathbf{V}_t\right)^{1-\eta}}_{\text{firm's payoff from total surplus}},$$

which (as stated) goes by the terminology “Nash maximand.” The (exogenous) bargaining power of the potential new employee is η, the (exogenous) bargaining power of the potential new employer is $1-\eta$, and $\eta\in(0,1)$.

In contrast to Nash bargaining, "proportional bargaining" contains no optimization whatsoever.[2] The asserted surplus-splitting condition for period-t proportionally-bargained wages is

$$\mathbf{W}(w_t)-\mathbf{U}_t=\left(\frac{\eta}{1-\eta}\right)\cdot\left(\mathbf{J}(w_t)-\mathbf{V}_t\right) \tag{5}$$

(NOTE: You may prefer to begin with parts c and d instead of parts a and b. If you choose that route and successfully work through parts c and d, the solutions to parts a and b are nested in the solutions to parts c and d. If you choose that route and get stuck, however, the analysis and solutions to parts a and b will (likely) help shed light on how to proceed in parts c and d.)

a. Suppose the exogenous separation rate of jobs is $\rho_x=1$. For the "full turnover" (i.e., $\rho_x=1$) case, compute the Nash-bargained surplus-splitting condition. **Display the Nash-bargained surplus-splitting condition in an intuitive, economically-informative manner; clearly display your final Nash surplus-splitting condition by drawing a box around it.**

   In the analysis leading to the final Nash surplus-splitting condition, clearly and carefully provide important algebraic steps/logic that lead to the solution**.** (No need to display each and every step of the algebra – it is left up to you to decide which algebraic steps are the most important one to display for, say, a referee that is reviewing your work.)

Solution: Regardless of whether or not $\rho_x=1$ or $0<\rho_x<1$, the Nash-bargained surplus-splitting condition is

$$\frac{\mathbf{W}_t-\mathbf{U}_t}{1-\tau_t^n}=\frac{\eta}{1-\eta}\mathbf{J}_t\,.$$

b. Based on the Nash surplus-splitting condition in part a, rewrite it to obtain the period-t real wage equation. **The final solution for the Nash-bargained period-t real wage equation should be expressed in the form** $w_t=....$ (the term in ellipsis **("…")** on the right-hand side is left to you to determine)**; clearly display your final Nash-bargained period-t real wage expression by drawing a box around it. (NOTE: The real wage expression may NOT include the terms $\mathbf{W}(w_t)$, $\mathbf{U}_t$, $\mathbf{J}(w_t)$, $\mathbf{V}_t$, $\forall t$.)**

   In the analysis leading to the final Nash-bargained period-t real wage expression, clearly and carefully provide important algebraic steps/logic that lead to the solution**.**

[2] The proportional bargaining solution was first developed by Kalai and Smorodinsky as an alternative to the Nash bargaining solution. (Kalai, Ehud and Meir Smorodinsky. 1975. "Other solutions to Nash's bargaining problem." *Econometrica*. Vol. 43 (3): p. 513–518.)

(No need to display each and every step of the algebra – it is left up to you to decide which algebraic steps are the most important one to display for, say, a referee that is reviewing your work.)

Solution: Using the parameter value $\rho_x = 1$ and substituting the period-t payoff expressions stated above into the Nash surplus-splitting condition in part a, a few steps of algebra lead to the period-t Nash real wage

$$w_t^{NASH} = \eta z_t + (1-\eta)\left(\frac{b}{1-\tau_t^n}\right).$$

c. Suppose the exogenous separation rate of jobs is $0 < \rho_x < 1$. For the "long-lasting employment relationships" (i.e., $0 < \rho_x < 1$) case, compute the Nash-bargained surplus-splitting condition. **Display the Nash-bargained surplus-splitting condition in an intuitive, economically-informative manner; clearly display your final Nash surplus-splitting condition by drawing a box around it.**

In the analysis leading to the final Nash surplus-splitting condition, clearly and carefully provide important algebraic steps/logic that lead to the solution**.** (No need to display each and every step of the algebra – it is left up to you to decide which algebraic steps are the most important one to display for, say, a referee that is reviewing your work.)

Solution: Regardless of whether or not $\rho_x = 1$ or $0 < \rho_x < 1$, the Nash-bargained surplus-splitting condition is

$$\frac{\mathbf{W}_t - \mathbf{U}_t}{1-\tau_t^n} = \frac{\eta}{1-\eta}\mathbf{J}_t.$$

d. Based on the Nash surplus-splitting condition in part c, rewrite it to obtain the period-t real wage equation. **The final solution for the Nash-bargained period-t real wage equation should be expressed in the form** $w_t = ...$ (the term in ellipsis **("…")** on the right-hand side is left to you to determine)**; clearly display your final Nash-bargained period-t real wage expression by drawing a box around it. (NOTE: The real wage expression may NOT include the terms $\mathbf{W}(w_t)$, $\mathbf{U}_t$, $\mathbf{J}(w_t)$, $\mathbf{V}_t$, $\forall t$.)**

In the analysis leading to the final Nash-bargained period-t real wage expression, clearly and carefully provide important algebraic steps/logic that lead to the solution**.** (No need to display each and every step of the algebra – it is left up to you to decide which algebraic steps are the most important one to display for, say, a referee that is reviewing your work.)

Solution: Substituting the period-t payoff expressions stated above (along with their period-(t+1) counterparts) into the Nash surplus-splitting condition in part c, several steps of algebra leads to the period-t Nash real wage

$$w_t^{NASH} = \eta z_t + (1-\eta)\left(\frac{b}{1-\tau_t^n}\right) + \eta(1-\rho_x)E_t\left\{\Xi_{t+1|t}\left[1-(1-k_{t+1}^h)\left(\frac{1-\tau_{t+1}^n}{1-\tau_t^n}\right)\right]\frac{\gamma}{k_{t+1}^f}\right\}.$$

The algebra behind this is admittedly tedious and/or challenging.[3] A big-picture takeaway economic observation from this wage expression, though, is that not only does the period-t labor income tax rate τ_t^n influence the period-t wage, the (in expectational terms) **change between period-t and period-(t+1) in the after tax term** $(1-\tau_t^n)$ also influences the period-t wage.

e. Suppose the exogenous separation rate of jobs is $\rho_x = 1$. Using the surplus-splitting condition for proportional bargaining stated above, rewrite it to obtain the period-t real wage equation. **The final solution for the proportionally-bargained period-t real wage equation should be expressed in the form** $w_t = ...$ (the term in ellipsis **("…")** on the right-hand side is left to you to determine)**; clearly display your final proportionally-bargained period-t real wage expression by drawing a box around it. (NOTE: The real wage expression may NOT include the terms $\mathbf{W}(w_t)$, $\mathbf{U}_t$, $\mathbf{J}(w_t)$, $\mathbf{V}_t$, $\forall t$.)**

   In the analysis leading to the final proportionally-bargained period-t real wage expression, clearly and carefully provide important algebraic steps/logic that lead to the solution**.** (No need to display each and every step of the algebra – it is left up to you to decide which algebraic steps are the most important one to display for, say, a referee that is reviewing your work.)

Solution: Using the parameter value $\rho_x = 1$ and substituting the period-t payoff expressions (1) and (2) into the proportionally-bargained surplus-splitting condition (5), a few steps of algebra leads to the period-t proportionally-bargained real wage

$$\begin{aligned} w_t^{PROP} &= \left(\frac{\eta}{1-(1-\eta)\cdot\tau_t^n}\right)\cdot z_t + \left(\frac{1-\eta}{1-(1-\eta)\cdot\tau_t^n}\right)\cdot b \\ &= \frac{\eta\cdot z_t + (1-\eta)\cdot b}{1-(1-\eta)\cdot\tau_t^n}. \end{aligned}$$

[3] For the intermediate steps of algebra, see Arseneau and Chugh (2012 *JPE*, starting from the bottom of p. 970 through the end of p. 971).

f. Suppose the exogenous separation rate of jobs is $0 < \rho_x < 1$. For the "long-lasting employment relationships" (i.e., $0 < \rho_x < 1$) case, rewrite the surplus-splitting condition for proportional bargaining stated above to obtain the period-t real wage equation. **The final solution for the proportionally-bargained period-t real wage equation should be expressed in the form** $w_t = ...$ (the term in ellipsis **("…")** on the right-hand side is left to you to determine)**; clearly display your final proportionally-bargained period-t real wage expression by drawing a box around it. (NOTE: The real wage expression may NOT include the terms $\mathbf{W}(w_t)$, $\mathbf{U}_t$, $\mathbf{J}(w_t)$, $\mathbf{V}_t$, $\forall t$.)**

In the analysis leading to the final proportionally-bargained period-t real wage, clearly and carefully provide important algebraic steps/logic that lead to the solution**.** (No need to display each and every step of the algebra – it is left up to you to decide which algebraic steps are the most important one to display for, say, a referee that is reviewing your work.)

Solution: Substituting the period-t payoff expressions stated above (along with their period-(t+1) counterparts) into the proportionally-bargained surplus-splitting condition (as also provided in the question prompt above), several steps of algebra leads to the period-t proportionally-bargained real wage.

In a bit more detail, the first step is to substitute the payoff expressions (1) and (2) into the proportionally-bargained surplus-splitting condition (5). Next, use the period-(t+1) surplus-splitting condition (5) to get

$$-b + (1-\tau_t^n)w_t + (1-\rho_x)E_t\left\{\Xi_{t+1|t}(1-k_{t+1}^h)\left(\frac{\eta}{1-\eta}\right)\mathbf{J}(w_{t+1})\right\} = \left(\frac{\eta}{1-\eta}\right)\mathbf{J}(w_t).$$

Using the Based on the payoff expression

Isolating terms in w_t on the left-hand side gives

$$w_t\left[1-\tau_t^n + \frac{\eta}{1-\eta}\right] = \left(\frac{\eta}{1-\eta}\right)z_t + b + \left(\frac{\eta}{1-\eta}\right)(1-\rho_x)E_t\left\{\Xi_{t+1}\frac{\gamma}{k_{t+1}^f}\right\} - \left(\frac{\eta}{1-\eta}\right)(1-\rho_x)E_t\left\{\Xi_{t+1}(1-k_{t+1}^h)\frac{\gamma}{k_{t+1}^f}\right\}$$

Collecting the expectational terms on the right-hand side gives

$$w_t\left[\frac{(1-\eta)(1-\tau_t^n)+\eta}{1-\eta}\right] = \left(\frac{\eta}{1-\eta}\right)z_t + b + \left(\frac{\eta}{1-\eta}\right)(1-\rho_x)E_t\left\{\Xi_{t+1}\frac{\gamma k_{t+1}^h}{k_{t+1}^f}\right\}.$$

For the Cobb-Douglas matching function, $\theta_{t+1} = \dfrac{k_{t+1}^h}{k_{t+1}^f}$, from which it follows that the period-t proportionally-bargained real wage is.

$$w_t^{PROP} = \frac{\eta z_t + (1-\eta)b}{1-(1-\eta)\tau_t^n} + \left(\frac{\eta}{1-(1-\eta)\tau_t^n}\right)(1-\rho_x)E_t\left\{\Xi_{t+1|t}\gamma\cdot\theta_{t+1}\right\}.$$

The following parts ask you to compare and contrast Nash-bargained wages with proportionally-bargained wage based on the results above.

(Note: For the grading process, it will be difficult, if not impossible, to carry errors in earlier parts of the question into the ensuing parts of the question.)

g. Comparing the Nash-bargained real wage expression from part d with the proportionally-bargained real wage expression from part f, is there any numerical value of the income tax rate τ_t^n that makes the two wages **identical** to each other? **Clearly** show/explain whether or not there is.

Solution: The only labor income tax rate for which both the Nash-bargained wage and the proportionally-bargained wage coincide is $\tau_t^n = 0$.

h. Suppose that the labor income tax rate is constant across periods – that is, $\tau_t^n = \bar{\tau}^n \ \forall t$. Express the Nash-bargained period-t real wage obtained in part d taking into account this time-invariant labor income tax rate. **Display your final solution clearly by drawing a box around it.**

Solution: With a time-invariant labor income tax rate, the Nash-bargained wage is

$$w_t^{NASH} = \eta z_t + (1-\eta)\left(\frac{b}{1-\bar{\tau}^n}\right) + \eta(1-\rho_x)E_t\left\{\Xi_{t+1|t}\gamma\cdot\theta_{t+1}\right\}.$$

i. If the labor tax rate were $\tau_t^n = \bar{\tau}^n \ \forall t$, is there any numerical value of the income tax rate τ_t^n that makes the two wages **identical** to each other? **Clearly** show/explain whether or not there is.

Solution: No, there is no value of $\bar{\tau}^n$ that will make the two wage functions identical.