The Baseline RBC Model: Theory and Computation

January 8, 2018

STYLIZED MACRO FACTS

- **Foundation of (virtually) all DSGE models (e.g., RBC model) is Solow growth model**
- **So want/need/desire business-cycle models to be consistent with basic growth facts**

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  1. **Output per worker exhibits ~constant growth**
  2. **Capital per worker exhibits ~constant growth**
  3. **Rate of return on capital is ~constant**
  4. **Capital-output ratio is ~constant**
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  - **Interested in fluctuations around long-run growth path**
- **How to construct/extract long-run trend?**
  - **Most common procedure: HP filter**
  - **Eliminates (if data were stationary) fluctuations at frequencies lower than eight years**

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  - **Most common procedure: HP filter**
  - **Eliminates (if data were stationary) fluctuations at frequencies lower than eight years**
  - **An alternative: band pass (BP) filter – allows specifying upper and lower frequencies to be eliminated**

Cooley volume, Chapter 1

STYLIZED MACRO FACTS

1947:Q1-1996:Q4

	Standard Deviation (%)	Relative Std Deviation	First-order Autocorr.	Corr(*X*,*GDP*)
GDP	**1.84**	**1**	**0.83**	**1**
PCE	**1.33**	**0.73**	**0.79**	**0.82**

STYLIZED MACRO FACTS

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	PCE	1.15	0.64	0.73	0.84

STYLIZED MACRO FACTS

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	PCE	1.15	0.64	0.73	0.84
1984:Q1-2006:Q4 "Great Moderation" Good Luck? Or Good Policy?	GDP	0.91	1	0.87	1
	PCE	0.78	0.85	0.83	0.86

STYLIZED MACRO FACTS

- **Kaldor's Stylized Facts:**
  1. **Output per worker exhibits ~constant growth**
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- **Some basic cyclical volatilities – SD% (i.e., time-series SD of HP-filtered component)**

GDP: 1.47% (1974Q1-2009Q4)	**C: 1.16% (1974Q1-2009Q4)** **CNDUR: 1.05%** **CSERV: 0.74%** **CDUR: 3.82%**
I: 7.03% (1974Q1-2009Q4)	**TOTAL HOURS: 1.59% (K&R)** **AVG HOURS: 0.63% (K&R)**
WAGE: 0.76% (K&R)	

WHAT DO WE WANT TO MODEL?

- Relative volatilities
- Persistence (i.e., first-order serial correlation) of various series
- Business cycle comovements
  - Corr(C, Y)
  - Corr(I, Y)
  - Corr(HOURS, Y)
  - Corr(WAGE, Y)
  - Others?...

- Labor markets?
  - Extensive margin – movements of individuals in and out of employment (i.e., work H = 0 hours or H > 0 hours)
  - Intensive margin – how many hours to work given an individual already works (i.e., work H = 39 hours or H = 40 hours or H = 41 etc…)
  - The basic RBC model blurs the difference
    - Prescott: "LS elasticity of 3 is right…"

The Three Macro Markets

- **Goods Market(s)**
- **Labor Market(s)**
- **Asset/Savings Market(s)**

- **Consumers**
  - **Demand goods**
  - **Supply labor**
  - **Supply assets/savings**
- **Firms**
  - **Produce goods**
  - **Demand labor**
  - **Demand assets/savings (capital)**
- **Government: auxiliary in the basic model**

Building Blocks

- **Consumers**
  - **Maximize lifetime utility (i.e., a dynamic problem)**
- **Firms**
  - **Maximize profits**
- **Prices adjust to clear all markets**
  - **Hence a general equilibrium model**
- **Unpredictable fluctuations in total factor productivity (TFP) are the driving source of business cycles in baseline RBC model**
  - **Identify TFP as the Solow residual**
    - $y(t) = z(t)*f(k(t), n(t))$

HOUSEHOLDS

- **Maximize lifetime utility (i.e., a dynamic problem) subject to sequence of budget constraints:**

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \quad \textbf{s.t.} \quad c_t + k_{t+1} - (1-\delta)k_t = w_t n_t + r_t k_t$$

- **Set up Lagrangian; optimality conditions**
  - **Consumption-Leisure Optimality Condition: MRS between consumption and labor equals real wage**

$$-\frac{u_n(c_t, n_t)}{u_c(c_t, n_t)} = w_t$$

  - **Consumption-Savings Optimality Condition (Euler equation): MRS between present and future consumption equals real return on savings (a difference equation)**

$$u_c(c_t, n_t) = \beta E_t \left\{ u_c(c_{t+1}, n_{t+1})(1 + r_{t+1} - \delta) \right\}$$

THE REST OF THE MODEL

- **Firms: maximize profits period-by-period** $\max_{n_t, k_t} z_t f(k_t, n_t) - w_t n_t - r_t k_t$
  - **FOCs yield factor-pricing conditions:** $w_t = z_t f_n(k_t, n_t), r_t = z_t f_k(k_t, n_t)$

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- **Government: omit from baseline RBC model**
- **Resource constraint:** $c_t + k_{t+1} - (1-\delta)k_t = z_t f(k_t, n_t)$

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- **Resource constraint:** $c_t + k_{t+1} - (1-\delta)k_t = z_t f(k_t,n_t)$
- **Exogenous process**

$$\ln z_{t+1} = (1-\rho_z)\ln \bar{z} + \rho_z \ln z_t + \varepsilon_{t+1}^z$$

  - **TFP follows AR(1) (satisfies Markov property), with persistence ρ_z**
  - **Average productivity zbar; white noise process $\varepsilon \sim N(0, \sigma^2_z)$**
  - **Specification in logs implies fluctuations are in deviations of TFP from the average zbar**

PUTTING THE MODEL TOGETHER: EQUILIBRIUM

INDIVIDUALS' DECISIONS ARE OPTIMAL

- **Consumer decisions:**
  - **Taking as given the real wage and the rental price of capital, choices of consumption, investment, and labor solve utility maximization**
- **Firm decisions:**
  - **Taking as given the real wage and the rental price of capital, choices of labor and capital solve profit maximization**

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ALL MARKETS CLEAR

- **Prices in goods, labor, and asset/savings markets adjust**
  - **Price of consumption normalized to one in every period**
  - **Prices (and thus decisions) depend on how TFP (and any other exogenous processes) evolves over time**

THE MODEL DETERMINES:

Allocations: **consumption, labor, savings/investment**
Prices: **real wage, rental rate of capital**

THE EQUATIONS AND VARIABLES

- **Equilibrium Conditions for $t = 0 \ldots \infty$**

$$-\frac{u_n(c_t, n_t)}{u_c(c_t, n_t)} = w_t$$

$$u_c(c_t, n_t) = \beta E_t \left\{ u_c(c_{t+1}, n_{t+1})(1 + r_{t+1} - \delta) \right\}$$

$$c_t + k_{t+1} - (1-\delta)k_t = z_t f(k_t, n_t)$$

$$w_t = z_t f_n(k_t, n_t), \quad r_t = z_t f_k(k_t, n_t)$$

$$\ln z_{t+1} = (1-\rho_z)\ln \overline{z} + \rho_z \ln z_t + \varepsilon^z_{t+1}$$

- **Exogenous variables (the inputs to the model):** $\left\{ z_t \right\}_{t=0}^{\infty}$
- **Endogenous variables (the outputs of the model):** $\left\{ c_t, n_t, k_{t+1}, w_t, r_t \right\}_{t=0}^{\infty}$
  - **Easy to express wage and rental rate as functions of *z*, *k*, and *n***

THE EQUATIONS AND VARIABLES

- **Equilibrium Conditions**

$$-\frac{u_n(c_t,n_t)}{u_c(c_t,n_t)} = z_t f_n(k_t,n_t)$$

Consumption-Labor Efficiency Condition

$$u_c(c_t,n_t) = \beta E_t \left\{ u_c(c_{t+1},n_{t+1})(1+z_{t+1} f_k(k_{t+1},n_{t+1}) - \delta) \right\}$$

Consumption-Investment Efficiency Condition

$$c_t + k_{t+1} - (1-\delta)k_t = z_t f(k_t,n_t)$$

Resource Constraint

$$\ln z_{t+1} = (1-\rho_z)\ln \overline{z} + \rho_z \ln z_t + \varepsilon^z_{t+1}$$

Law of motion for TFP

- **Exogenous variables (the inputs to the model):** $\left\{ z_t \right\}_{t=0}^{\infty}$
- **Endogenous variables (the outputs of the model):** $\left\{ c_t, n_t, k_{t+1} \right\}_{t=0}^{\infty}$

HOW TO USE THE MODEL?

- **Need to specify/solve for FUNCTIONS (aka "decision rules") that describe how:**
  - **Consumers make choices based on prices and policies**
  - **Firms make choices based on prices and policies**
  - **Prices depend on state variables (capital, TFP, and all other exogenous variables)**

- **Except for very special cases, must turn to quantitative (i.e., numerical) methods**
  - **Because of the difference (differential) equation in the model:**

Euler equation

APPROXIMATIONS

- **Looking for an equilibrium in which endogenous variables are time-invariant functions of the state of the model** $\mathbf{S}_t \equiv [k_t; z_t]$
  - **State describes the dynamic position of the model**
  - **So looking for** $c(\mathrm{S_t}), n(\mathrm{S_t}), k(\mathrm{S_t})$

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- **Cannot solve difference equations analytically in general**
  - **These solutions are unknowable in general**
  - **Hence need to approximate – so look for**

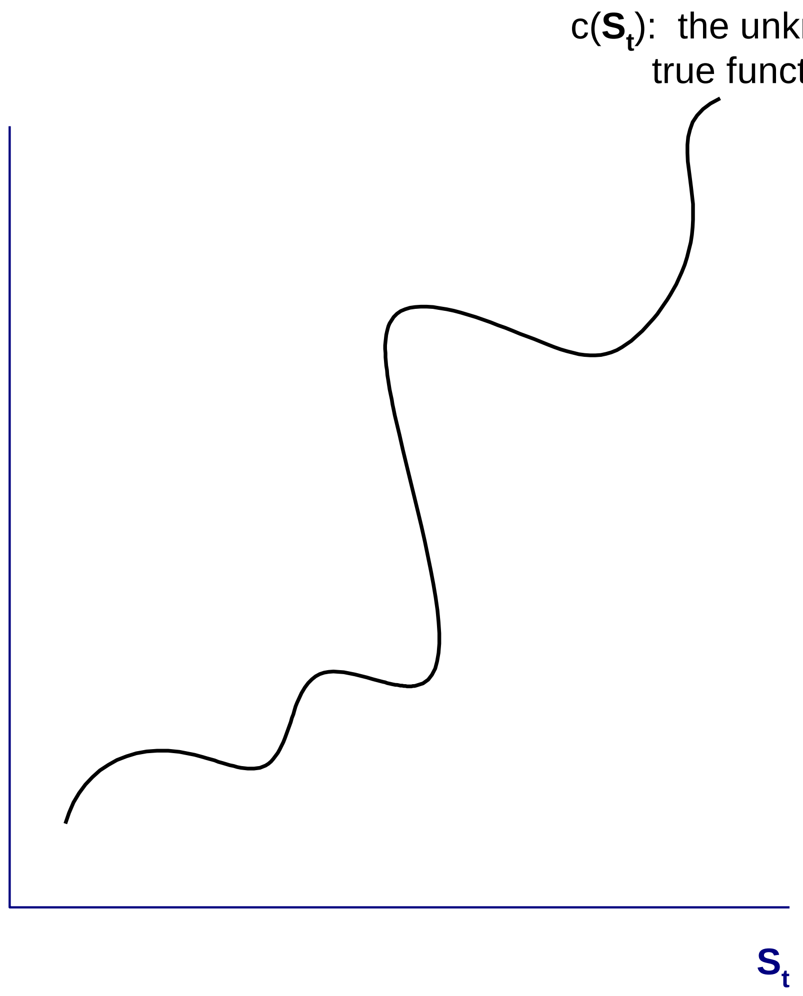
$$c^{approx}(\mathrm{S}_t), n^{approx}(\mathrm{S}_t), k^{approx}(\mathrm{S}_t)$$

which are hopefully near the (unknowable…) truth…

TYPES OF APPROXIMATIONS

- **Global:** approximated functions are close to true functions "everywhere" (over a very broad range of states)
  - Hard to implement for medium- and large-scale models given current hardware capacity
  - Several popular methods
    - Chebyshev polynomials
    - Finite-element methods
    - Value function iteration
- **Local:** approximated functions are close to true functions only in a relatively small range of the state space
  - Much easier to implement
  - Based on Taylor approximations
    - Linear (first-order)
    - Quadratic (second-order)
    - Etc.

APPROXIMATIONS



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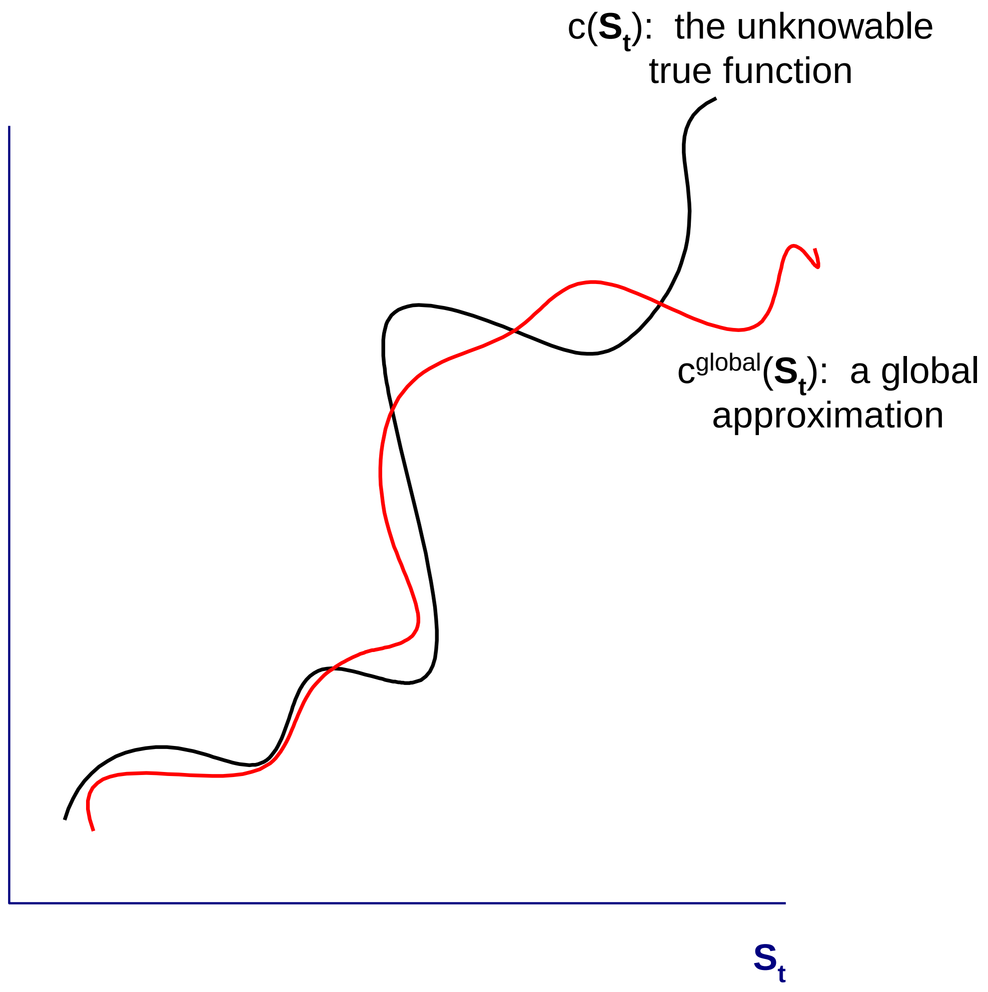

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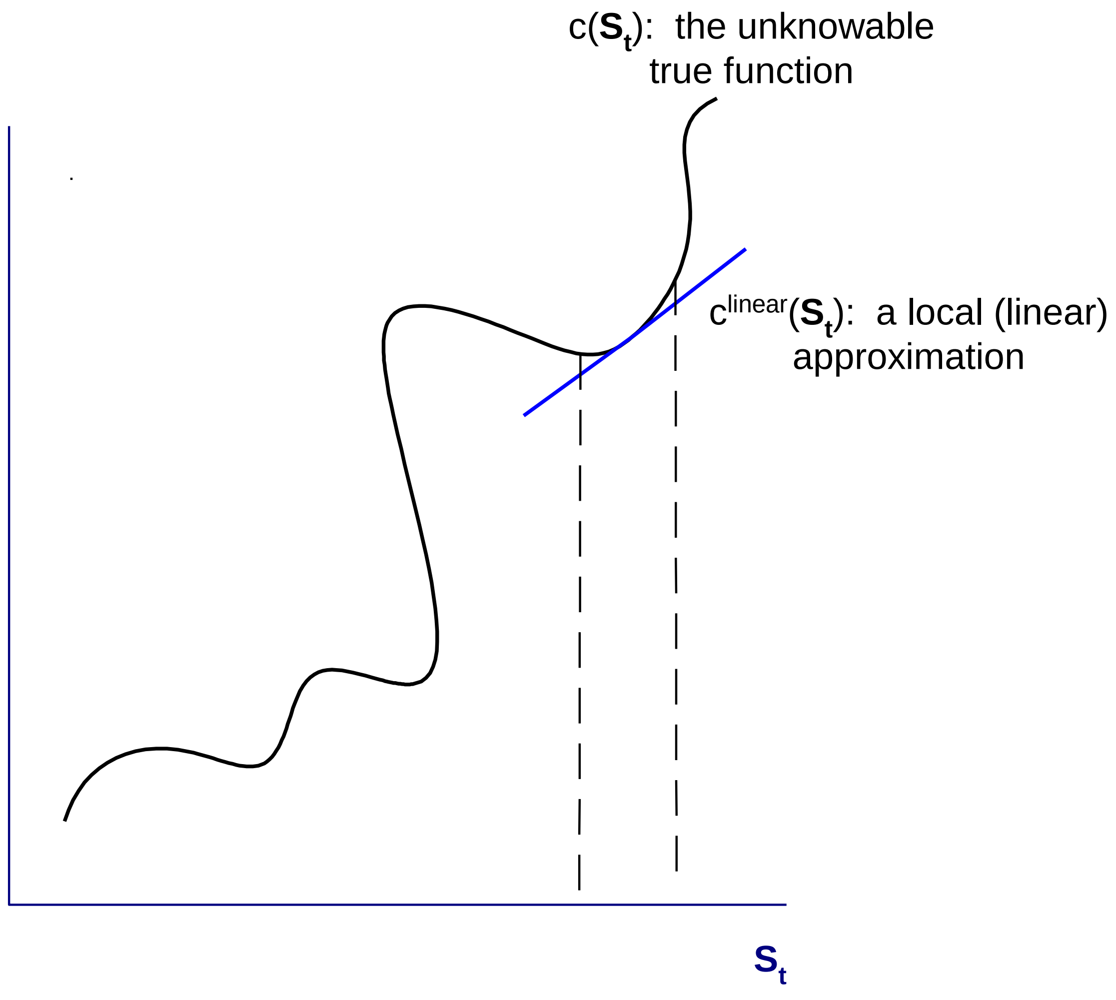

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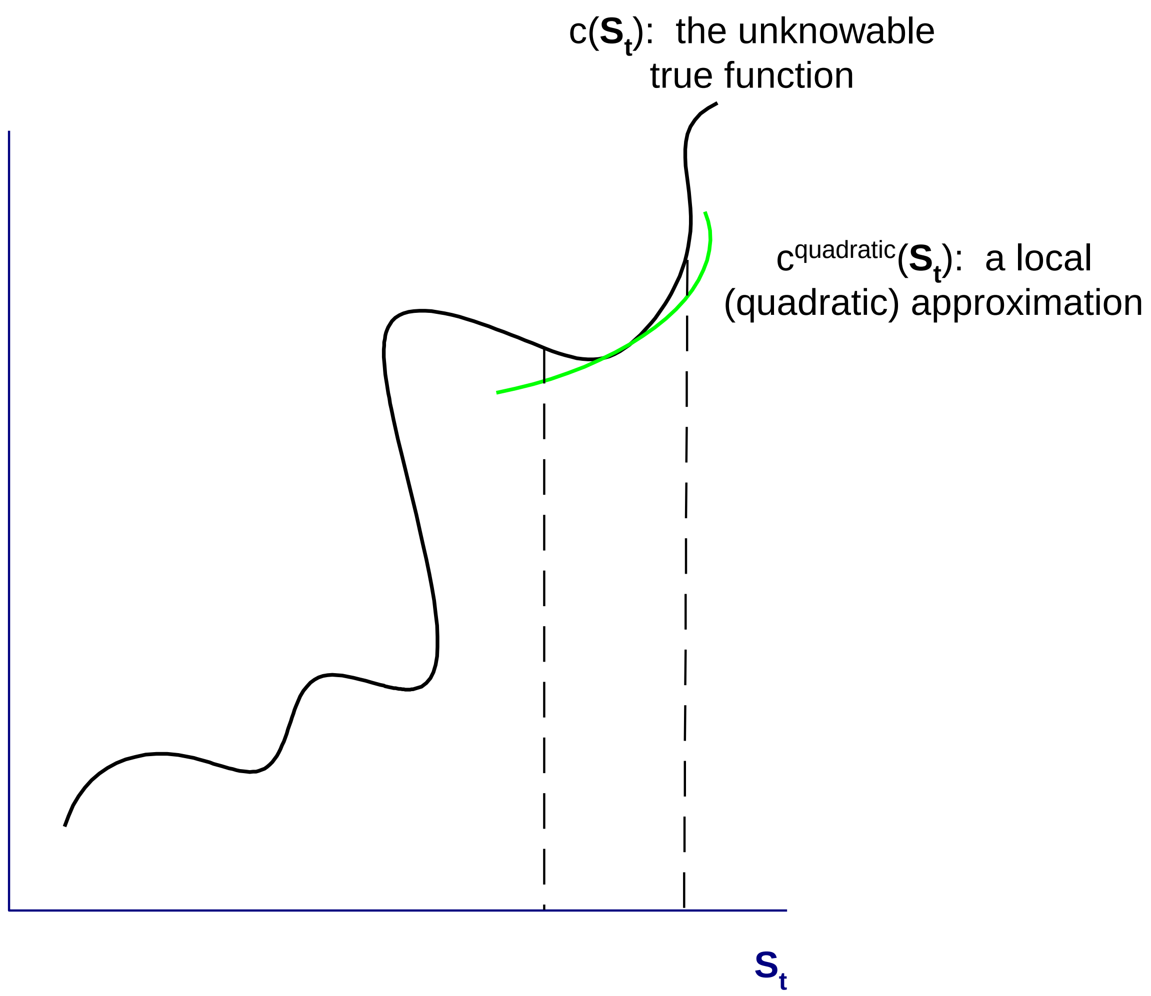

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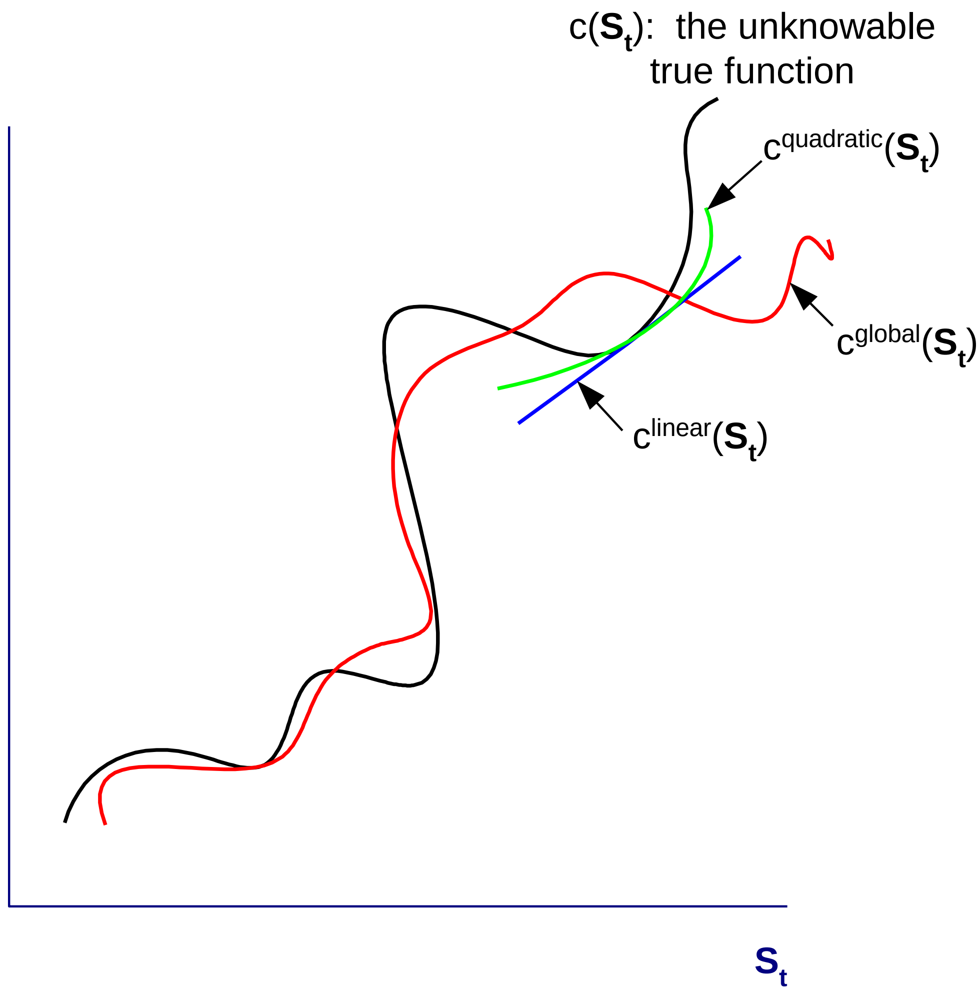

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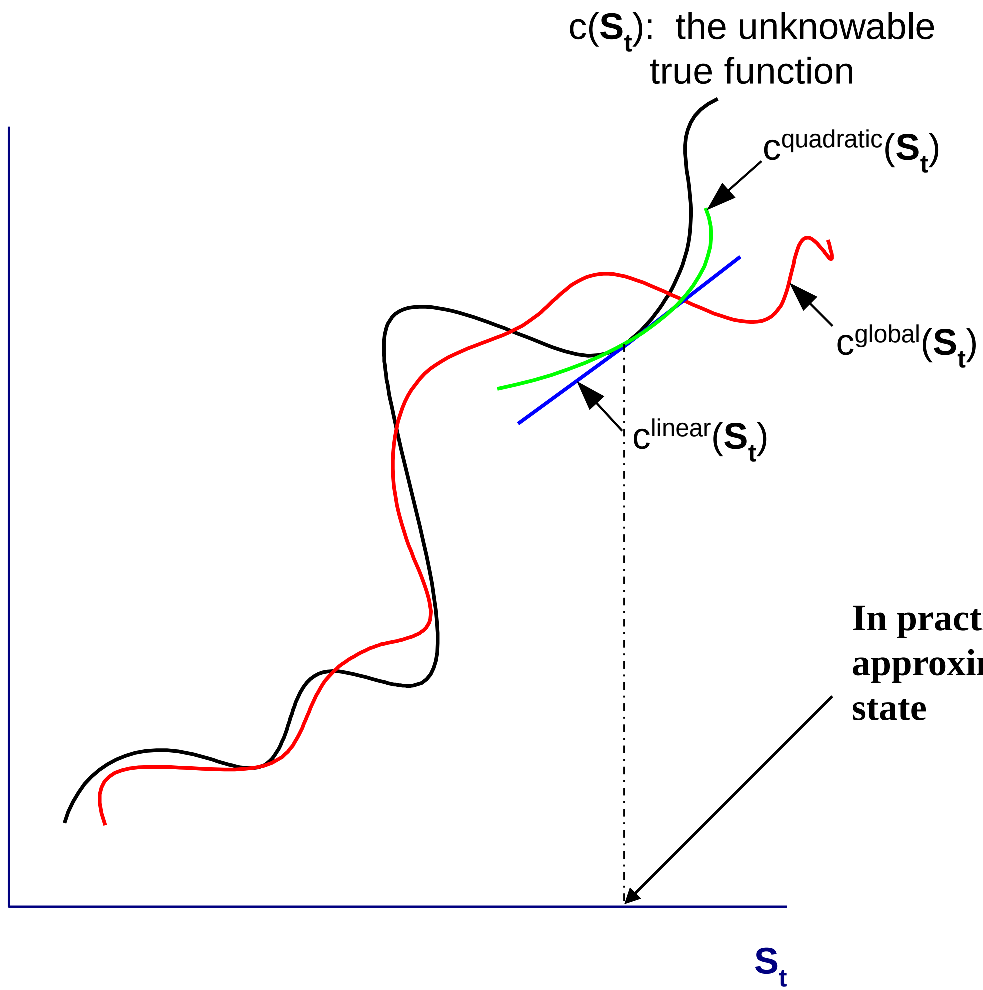

In practice, conduct local approximation around steady state

STEADY STATE

- **Shut down all shocks and set exogenous variables at their means**
- **Let model economy run for many (infinite) periods**
  - **Time eventually "doesn't matter" any more**
  - **Drop all time indices**

$$-\frac{u_n(c,n)}{u_c(c,n)} = \bar{z}f_n(k,n)$$

$$u_c(c,n) = \beta u_c(c,n)(1+\bar{z}f_k(k,n)-\delta)$$

$$c + \delta k = \bar{z}f(k,n)$$

- **(*c*, *n*, *k*) is a triple of scalars that are the deterministic steady state (aka long run) outcomes of the model economy**

- **Given functional forms and parameter values, solve for (*c*, *n*, *k*)**
  - **Conduct local approximation around this point**

Linear Approximation of the Baseline RBC Model

January 8, 2018

LINEARIZATION

- **For** $f(x, y, z) = 0$**, multivariable Taylor linear expansion around** $(\bar{x}, \bar{y}, \bar{z})$

$$f(x, y, z) \approx f(\bar{x}, \bar{y}, \bar{z}) + f_x(\bar{x}, \bar{y}, \bar{z})(x - \bar{x}) + f_y(\bar{x}, \bar{y}, \bar{z})(y - \bar{y}) + f_z(\bar{x}, \bar{y}, \bar{z})(z - \bar{z})$$

(Illustrative example in scalars)

LINEARIZATION OF THE RBC MODEL

- **For $f(x,y,z)=0$, multivariable Taylor linear expansion around $(\bar{x},\bar{y},\bar{z})$**

$$f(x,y,z)\approx f(\bar{x},\bar{y},\bar{z})+f_x(\bar{x},\bar{y},\bar{z})(x-\bar{x})+f_y(\bar{x},\bar{y},\bar{z})(y-\bar{y})+f_z(\bar{x},\bar{y},\bar{z})(z-\bar{z})$$

- **Four equations describe the dynamic solution to RBC model**

(Illustrative example in scalars)

  - **Consumption-leisure efficiency condition**

$$-\frac{u_n(c_t,n_t)}{u_c(c_t,n_t)}=z_t m_n(k_t,n_t)$$

  - **Consumption-investment efficiency condition**

$$u_c(c_t,n_t)=\beta E_t\left[u_c(c_{t+1},n_{t+1})\left(1-\delta+z_{t+1}m_k(k_{t+1},n_{t+1})\right)\right]$$

  - **Aggregate resource constraint**

$$c_t+k_{t+1}-(1-\delta)k_t=z_t m(k_t,n_t)$$

  - **Law of motion for TFP**

$$\ln z_{t+1}=(1-\rho_z)\ln\bar{z}+\rho_z\ln z_t+\varepsilon^z_{t+1}$$

STEADY STATE

- **Deterministic steady state the natural local point of approximation**
- **Shut down all shocks and set exogenous variables at their means**
- **The Idea: Let economy run for many (infinite) periods**
  - **Time eventually "doesn't matter" any more**
  - **Drop all time indices**

$$-\frac{u_n(\bar{c},\bar{n})}{u_c(\bar{c},\bar{n})} = \bar{z}m_n(\bar{k},\bar{n})$$

$$u_c(\bar{c},\bar{n}) = \beta u_c(\bar{c},\bar{n})\left[m_k(\bar{k},\bar{n}) + 1 - \delta\right]$$

$$\bar{c} + \delta\bar{k} = \bar{z}m(\bar{k},\bar{n})$$

$$\ln\bar{z} = (1-\rho_z)\ln\bar{z} + \rho_z\ln\bar{z} \Rightarrow \bar{z} = \bar{z} \quad \text{(a parameter of the model)}$$

- **Given functional forms and parameter values, solve for (*c*, *n*, *k*)**
  - **The steady state of the model**
  - **Taylor expansion around this point**

LINEARIZATION ALGORITHMS

- **Schmitt-Grohe and Uribe (2004 *JEDC*)**
  - **A perturbation algorithm**
    - **A class of methods used to find an approximate solution to a problem that cannot be solved exactly, by starting from the exact solution of a related problem**
    - **Applicable if the problem can be formulated by adding a "small" term to the description of the exactly-solvable problem**
  - **(Matlab code available through Columbia Dept. of Economics web site – DO NOT USE IN THIS CLASS!)**

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- **Uhlig (1999, chapter in *Computational Methods for the Study of Dynamic Economies*)**
  - **Uses a generalized eigen-decomposition**
    - **Typically implemented with Schur decomposition (Sims algorithm)**
  - **Matlab code available at**

    http://www2.wiwi.hu-berlin.de/institute/wpol/html/toolkit.htm

LINEARIZATION OF THE RBC MODEL

Define co-state vector and state vector

$$y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix} \qquad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$

LINEARIZATION OF THE RBC MODEL

Define co-state vector and state vector

$$y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix} \qquad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$

Order model's dynamic equations in a vector $\equiv f(y_{t+1}, y_t, x_{t+1}, x_t) = 0$

Consumption-leisure efficiency condition

Consumption-investment efficiency condition

Aggregate resource constraint

Law of motion for TFP

LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

1. **Differentiate** $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** y_{t+1}

First derivatives with respect to:

	c_{t+1}	n_{t+1}
Consumption-leisure efficiency condition		
Consumption-investment efficiency condition		
Aggregate resource constraint		
Law of motion for TFP		

$= f_{y_{t+1}}$

LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

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First derivatives with respect to:

	c_t	n_t
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Consumption-investment efficiency condition		
Aggregate resource constraint		
Law of motion for TFP		

$= f_{y_t}$

LINEARIZATION OF THE RBC MODEL

<u>Need four matrices of derivatives</u>

3. Differentiate $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** $\boldsymbol{x_{t+1}}$

First derivatives with respect to:

	k_{t+1}	z_{t+1}
Consumption-leisure efficiency condition		
Consumption-investment efficiency condition		
Aggregate resource constraint		
Law of motion for TFP		

$= f_{x_{t+1}}$

LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

4. Differentiate $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** $\boldsymbol{x_t}$

First derivatives with respect to:

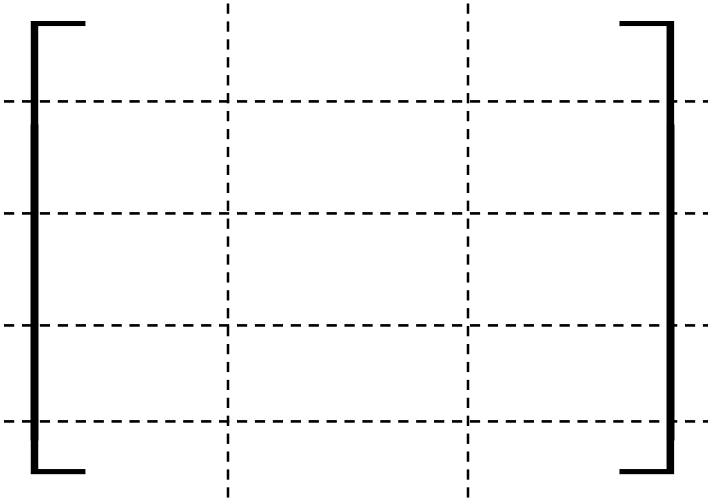
k_t z_t

Consumption-leisure efficiency condition

Consumption-investment efficiency condition

Aggregate resource constraint

Law of motion for TFP



$= f_{x_t}$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t)\right] = E_t\begin{bmatrix} f^1(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^2(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^3(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^4(y_{t+1}, y_t, x_{t+1}, x_t) \end{bmatrix}$$

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- f^1: **Consumption-leisure efficiency condition**
- f^2: **Consumption-investment efficiency condition**
- f^3: **Aggregate resource constraint**
- f^4: **Law of motion for TFP**

Conjecture equilibrium decision rules

Note: g(.) and h(.) are time invariant functions!

$$y_t = g(x_t,\sigma)$$
$$x_{t+1} = h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1}$$

"Perturbation parameter": governs size of shocks (→ σ)

Matrix of standard deviations of state variables (→ η)

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]=E_t\begin{bmatrix} f^1(y_{t+1},y_t,x_{t+1},x_t) \\ f^2(y_{t+1},y_t,x_{t+1},x_t) \\ f^3(y_{t+1},y_t,x_{t+1},x_t) \\ f^4(y_{t+1},y_t,x_{t+1},x_t) \end{bmatrix}$$

Consumption-leisure efficiency condition

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"Perturbation parameter": governs size of shocks

Matrix of standard deviations of state variables

Substitute decision rules into dynamic equations

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]=0$$

$$=E_t\left[f(g(x_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$

$$=E_t\left[f(g(h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$

$$\equiv F(x_t,\sigma)$$

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$$\equiv F(x_t,\sigma)$$

$$\Downarrow$$

$$F_x(x_t,\sigma) = 0 \qquad F_\sigma(x_t,\sigma) = 0$$

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Using chain rule and suppressing arguments

$$F_x(x_t,\sigma)=$$

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$\Downarrow$ **Using chain rule and suppressing arguments**

$$F_x(x_t,\sigma) = f_{y_{t+1}} \cdot g_x \cdot h_x$$

LINEARIZATION OF THE RBC MODEL

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$$\equiv F(x_t,\sigma)$$

$$\Downarrow$$

Using chain rule and suppressing arguments

$$F_x(x_t,\sigma) = f_{y_{t+1}} \cdot g_x \cdot h_x + f_{y_t} \cdot g_x$$

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$$\equiv F(x_t,\sigma)$$

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Using chain rule and suppressing arguments

$$F_x(x_t,\sigma) = f_{y_{t+1}} \cdot g_x \cdot h_x + f_{y_t} \cdot g_x + f_{x_{t+1}} \cdot h_x + f_{x_t}$$

$$= 0$$

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$$\equiv F(x_t,\sigma)$$

$\Downarrow$ **Using chain rule and suppressing arguments**

$$F_x(x_t,\sigma)=f_{y_{t+1}}\cdot g_x\cdot h_x+f_{y_t}\cdot g_x+f_{x_{t+1}}\cdot h_x+f_{x_t}$$
$$=0$$

Setting σ = 0 shuts down shocks

Holds, in particular, at the deterministic steady state $(\bar{x},0)$

$$F_x(\bar{x},0)=f_{y_{t+1}}\cdot g_x\cdot h_x+f_{y_t}\cdot g_x+f_{x_{t+1}}\cdot h_x+f_x=0$$

Each term is evaluated at the steady state – just as Taylor theorem requires

LINEARIZATION OF THE RBC MODEL

- A quadratic equation in the elements of g_x and h_x evaluated at the steady state

$$F_x(\bar{x},0) = f_{y_{t+1}}(\bar{x},0)\cdot g_x(\bar{x},0)\cdot h_x(\bar{x},0) + f_{y_t}(\bar{x},0)\cdot g_x(\bar{x},0) + f_{x_{t+1}}(\bar{x},0)\cdot h_x(\bar{x},0) + f_{x_t}(\bar{x},0) = 0$$

- Solve numerically for the elements of g_x and h_x (use `fsolve` OR conduct an eigenvalue decomposition in Matlab)

LINEARIZATION OF THE RBC MODEL

- **A quadratic equation in the elements of g_x and h_x evaluated at the steady state**

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- **Solve numerically for the elements of g_x and h_x (use `fsolve` OR conduct an eigenvalue decomposition in Matlab)**
- **Recall conjectured equilibrium decision rules**

$$y_t = g(x_t,\sigma)$$

$$x_{t+1} = h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}$$

LINEARIZATION OF THE RBC MODEL

- **A quadratic equation in the elements of g_x and h_x evaluated at the steady state**

$$F_x(\bar{x},0)=f_{y_{t+1}}(\bar{x},0)\cdot g_x(\bar{x},0)\cdot h_x(\bar{x},0)+f_{y_t}(\bar{x},0)\cdot g_x(\bar{x},0)+f_{x_{t+1}}(\bar{x},0)\cdot h_x(\bar{x},0)+f_{x_t}(\bar{x},0)=0$$

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$$y_t=g(x_t,\sigma)$$
$$x_{t+1}=h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1}$$

- **First-order approximation is**

$$y_t=g(x_t,\sigma)\approx g(\bar{x},0)+g_x(\bar{x},0)(x_t-\bar{x})+g_\sigma(\bar{x},0)\sigma$$
$$x_{t+1}=h(x_t,\sigma)\approx h(\bar{x},0)+h_x(\bar{x},0)(x_t-\bar{x})+h_\sigma(\bar{x},0)\sigma$$

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$$F_x(\bar{x},0) = f_{y_{t+1}}(\bar{x},0)\cdot g_x(\bar{x},0)\cdot h_x(\bar{x},0) + f_{y_t}(\bar{x},0)\cdot g_x(\bar{x},0) + f_{x_{t+1}}(\bar{x},0)\cdot h_x(\bar{x},0) + f_{x_t}(\bar{x},0) = 0$$

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- First-order approximation is

= 0

$$y_t = g(x_t,\sigma) \approx g(\bar{x},0) + g_x(\bar{x},0)(x_t - \bar{x}) + g_\sigma(\bar{x},0)\sigma$$
$$x_{t+1} = h(x_t,\sigma) \approx h(\bar{x},0) + h_x(\bar{x},0)(x_t - \bar{x}) + h_\sigma(\bar{x},0)\sigma$$

= 0

SGU Theorem 1:

g_σ = 0 and h_σ = 0

LINEARIZATION OF THE RBC MODEL

- A quadratic equation in the elements of g_x and h_x evaluated at the steady state

$$F_x(\bar{x},0) = f_{y_{t+1}}(\bar{x},0)\cdot g_x(\bar{x},0)\cdot h_x(\bar{x},0) + f_{y_t}(\bar{x},0)\cdot g_x(\bar{x},0) + f_{x_{t+1}}(\bar{x},0)\cdot h_x(\bar{x},0) + f_{x_t}(\bar{x},0) = 0$$

- Solve numerically for the elements of g_x and h_x (use `fsolve` OR conduct an eigenvalue decomposition in Matlab)
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$$y_t = g(x_t,\sigma)$$
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= 0

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= 0

SGU Theorem 1: $g_\sigma = 0$ and $h_\sigma = 0$

- DONE!!!
  - Now conduct impulse responses, tabulate business cycle moments, write paper

CERTAINTY EQUIVALENCE

- **Displayed by a model if decision rules do not depend on the standard deviation of exogenous uncertainty**
- **For stochastic problems with quadratic objective function and linear constraints, the decision rules are identical to those of the nonstochastic problem**

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- **For stochastic problems with quadratic objective function and linear constraints, the decision rules are identical to those of the nonstochastic problem**

- **Here, we have**

$$y_t = g(x_t, \sigma) \approx g(\bar{x}, 0) + g_x(\bar{x}, 0)(x_t - \bar{x}) + \underbrace{g_\sigma(\bar{x}, 0)}_{=0}\sigma$$

$$x_{t+1} = h(x_t, \sigma) \approx h(\bar{x}, 0) + h_x(\bar{x}, 0)(x_t - \bar{x}) + \underbrace{h_\sigma(\bar{x}, 0)}_{=0}\sigma$$

- **SGU Theorem 1: $g_\sigma = 0$ and $h_\sigma = 0$**
  - **First-order approximated decision rules do not depend on the size of the shocks, which is governed by σ**
  - **Not the same thing as "exact CE," but refer to it as CE**

LINEARIZING THE RBC MODEL

- **Assume** $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$

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- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ **)**

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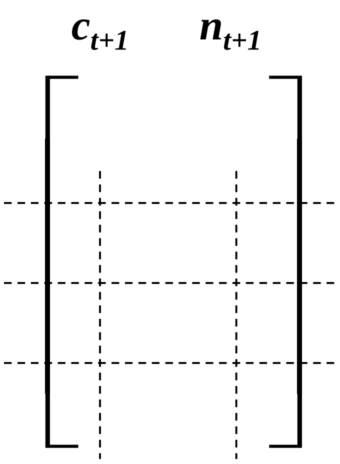
- Compute first row of matrix f_{yt+1}

Consumption-leisure efficiency condition

Consumption-investment efficiency condition

Aggregate resource constraint

Law of motion for TFP



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- **Compute first row of matrix f_{yt+1}**

	c_{t+1}	n_{t+1}
Consumption-leisure efficiency condition	0	
Consumption-investment efficiency condition		
Aggregate resource constraint		
Law of motion for TFP		

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- **Compute first row of matrix f_{yt}**

	c_t	n_t
Consumption-leisure efficiency condition	$\dfrac{\psi}{n_t}$	$-\dfrac{\psi c_t}{n_t^2} + \alpha(1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha-1}$
Consumption-investment efficiency condition		
Aggregate resource constraint		
Law of motion for TFP		

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- **Compute first row of matrix f_{xt+1}**

Consumption-leisure efficiency condition

Consumption-investment efficiency condition

Aggregate resource constraint

Law of motion for TFP

$$\begin{matrix} k_{t+1} & z_{t+1} \end{matrix}$$

$$\begin{bmatrix} 0 & 0 \\ & \\ & \\ & \end{bmatrix}$$

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- **Compute first row of matrix** ***f_{xt}***

	k_t	z_t
Consumption-leisure efficiency condition	$-\alpha(1-\alpha) z_t \dfrac{k_t^{\alpha-1}}{n_t^{\alpha}}$	$-(1-\alpha)\dfrac{k_t^{\alpha}}{n_t^{\alpha+1}}$
Consumption-investment efficiency condition		
Aggregate resource constraint		
Law of motion for TFP		

LINEARIZING THE RBC MODEL

- **In deterministic steady state, the first rows of f_{yt+1}, f_{yt}, f_{xt+1}, f_{xt} are**

f_{yt+1}	0	0
f_{yt}	$\frac{\psi}{\bar{n}}$	$-\frac{\psi\bar{c}}{\bar{n}^2}+\alpha(1-\alpha)\bar{z}\bar{k}^{\alpha}\bar{n}^{-\alpha-1}$
f_{xt+1}	0	0
f_{xt}	$-\alpha(1-\alpha)\bar{z}\frac{\bar{k}^{\alpha-1}}{\bar{n}^{\alpha}}$	$-(1-\alpha)\frac{\bar{k}^{\alpha}}{\bar{n}^{\alpha+1}}$

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f_{yt}	$\dfrac{\psi}{\bar{n}}$	$-\dfrac{\psi\bar{c}}{\bar{n}^2}+\alpha(1-\alpha)\bar{z}\bar{k}^{\alpha}\bar{n}^{-\alpha-1}$
f_{xt+1}	0	0
f_{xt}	$-\alpha(1-\alpha)\bar{z}\dfrac{\bar{k}^{\alpha-1}}{\bar{n}^{\alpha}}$	$-(1-\alpha)\dfrac{\bar{k}^{\alpha}}{\bar{n}^{\alpha+1}}$

- **How to compute derivatives f_{yt+1}, f_{yt}, f_{xt+1}, f_{xt}?**
  - **By hand (feasible for small models)**
  - **Schmitt-Grohe and Uribe Matlab analytical routines**
  - **Your own Maple or Mathematica programs**
  - **MuPad**
  - **Dynare package**

CALIBRATION?

- **Solving for the steady state?**
- **Choosing parameter values?**
- **Next: calibration of the baseline representative-agent (RBC + growth) model**