
BASICS OF DYNAMIC STOCHASTIC (GENERAL) EQUILIBRIUM

MARCH 2, 2017

HETEROGENEITY

- ❑ **Implementing representative consumer**
 - ❑ **An infinity of consumers, each indexed by a point on the unit interval [0,1]**
 - ❑ **Each individual is identical in preferences and endowments**
 - ❑ **Implies aggregate consumption demand and asset demand**

$$\text{Aggregate consumption demand} = \text{One individual's consumption demand} \times \mathbf{1}$$

$$\text{Aggregate savings demand} = \text{One individual's savings demand} \times \mathbf{1}$$

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- ❑ **Under some particular types of heterogeneity, a representative-consumer foundation of aggregates exists**
 - ❑ **Provided complete set of Arrow-Debreu securities exists...**
 - ❑ **...to allow individuals to diversify away (insure) their idiosyncratic risk**
- ❑ **Consider heterogeneity**
 - ❑ **In income realizations (from Markov process)**
 - ❑ **In initial asset holdings a**
 - ❑ **In utility functions (application to CRRA utility)**
 - ❑ **Example: two types of individuals to illustrate**

HETEROGENEITY

- **Two types of individuals, $i \in \{1,2\}$, each with population weight 0.5**

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u^i(c_t^i) \quad \text{subject to} \quad c_t^i + \sum_j R_t^j a_t^{ij} = y_t^i + a_{t-1}^i$$

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- Optimization between period t and state j in period $t+1$ (conditional on period t outcomes)

p^j denotes conditional probability of state j being realized in $t+1$

$$\frac{u_c^1(c_t^1)}{\beta u_c^1(c_{t+1}^{1j})} = \frac{p_{t+1}^j}{R_t^j}$$

$$\frac{p_{t+1}^j}{R_t^j} = \frac{u_c^2(c_t^2)}{\beta u_c^2(c_{t+1}^{2j})}$$

Given all individuals base choices on same prices and probabilities

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In all states at all dates

- PERFECT RISK SHARING**
 - IMRS, for each state j , equated across individuals**
 - Individuals experiencing **idiosyncratic** shocks can insure them away (provided complete markets)
- Risk sharing about equalizing **fluctuations** of $u'(\cdot)$ across individuals
 - Not about equalizing **levels** of $u'(\cdot)$ or consumption over time

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- If** initial conditions, period-zero outcomes, and $u(\cdot)$ are identical (e.g., due to identical a_0 and realized y_0), **then** risk sharing \rightarrow **identical** outcomes for all t
 \rightarrow **A representative consumer**

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Risk sharing across individuals \approx consumption smoothing for a given individual
 (= if initial conditions, $t=0$ outcomes, and $u(\cdot)$ identical)

RISK SHARING

- **Example: CRRRA utility, but heterogenous RRA/IES**

- $\sigma^1 \neq \sigma^2$

$$\left(\frac{c_t^1}{c_{t+1}^1} \right)^{-\sigma^1} = \left(\frac{c_t^2}{c_{t+1}^2} \right)^{-\sigma^2}$$

Perfect risk sharing

- **IMRS equated across individuals**

- **Growth rates of consumption **not** equated unless $\sigma^1 = \sigma^2$**

$$\frac{c_{t+1}^{1j}}{c_t^1} = \left(\frac{c_{t+1}^{2j}}{c_t^2} \right)^{\sigma^2/\sigma^1}$$

AGGREGATION

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- **Allocations are **Pareto-optimal** (implied by First Welfare Theorem)**

- **All MRS' s (across individuals, states, and dates) are equated**
 - **Even though **levels** of consumption may differ across individuals**
 - **No individual can be made better off without making some agent worse off**
 - **(Pareto welfare concept takes distributions of outcomes as given)**
 - **Due to complete financial markets**

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□ Pareto-optimal allocations + heterogeneity of utility functions

- There exists a utility function $u(c)$ in aggregate $c = c^1 + c^2$ that leads to the same aggregates (Constantanides (1982)); **if CRR, $u(\cdot)$ has $\sigma \in (\sigma^1, \sigma^2)$**

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- There exists a utility function $u(c)$ in aggregate $c = c^1 + c^2$ that leads to the same aggregates; **proof relies on general equilibrium theory**

AGGREGATION

- Now consider **economy-wide aggregates**

$$c_t = 0.5c_t^1 + 0.5c_t^2$$

$$y_t = 0.5y_t^1 + 0.5y_t^2$$

Aggregate consumption

**Aggregate income
(endowment)**

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(For each type of asset)

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- So far have been considering assets as claims (paper!) (partial equilibrium)
- **In aggregate, must be some tangible asset(s) backing them (gen. equil.)**
- No physical assets in model so far → $a_t = \underline{0 \text{ in aggregate}}$ for all t !

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$$0 = a_t = 0.5a_t^1 + 0.5a_t^2$$

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Aggregate assets = 0 if
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- In aggregate, must be some tangible asset(s) backing them (gen. equil.)**
- No physical assets in model so far → **$a_t = 0$ in aggregate for all t !**
- Heterogeneous individuals creating/buying/selling assets vis-à-vis each other
- Richer models
 - Mediate through “banking” or “insurance” markets, etc.
 - But only meaningful if some friction/imperfections in model of financial markets...
 - ...otherwise identical outcomes (in which case “banking” sector is a “veil”)

AGGREGATION

□ Economy-wide aggregates

$$c_t = 0.5c_t^1 + 0.5c_t^2$$

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Asset market clearing condition
(for each type of asset)

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□ Aggregate together two types' budget constraints

$$c_t^1 + \sum_j R_t^j a_t^{1j} = y_t^1 + a_{t-1}^1 \quad c_t^2 + \sum_j R_t^j a_t^{2j} = y_t^2 + a_{t-1}^2$$

□ **Weight by share of population**

□ **Impose asset-market clearing condition(s)**

$$\Rightarrow 0.5(c_t^1 + c_t^2) + \sum_j R_t^j \underbrace{0.5(a_t^{1j} + a_t^{2j})}_{= 0 \text{ across } j} = 0.5(y_t^1 + y_t^2) + 0.5 \underbrace{(a_{t-1}^1 + a_{t-1}^2)}_{= 0}$$

AGGREGATION

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$$\Rightarrow c_t = y_t$$

Goods market clearing condition – aka resource constraint

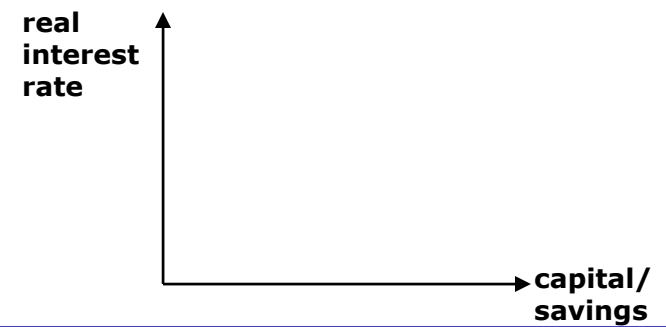
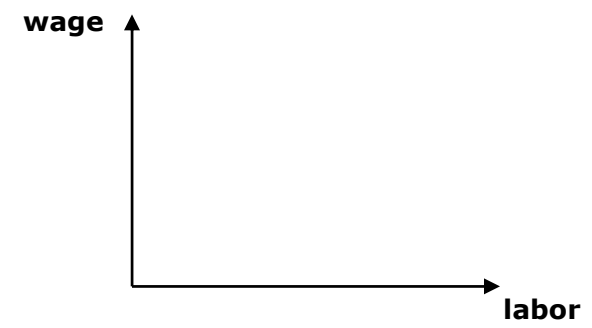
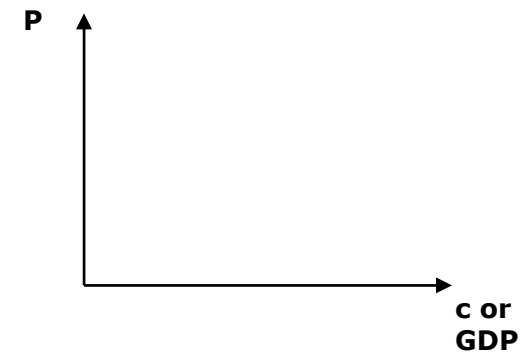
A general procedure for constructing economy-wide resource constraint goods available = goods used

THE THREE MACRO (AGGREGATE) MARKETS

☐ **Goods Markets**

☐ **Labor Markets**

☐ **Capital/Savings/Funds/Asset Markets
(aka Financial Markets)**



TOWARDS DYNAMIC GENERAL EQUILIBRIUM

- ❑ Lifecycle/permanent income consumption model the most basic building block of all macro models
- ❑ **Dynamic stochastic general equilibrium (DSGE) theory**
 - ❑ (DGE if deterministic)
 - ❑ **GE: simultaneous determination of prices and quantities in all markets (macro markets: goods, labor, capital)**
- ❑ **Foundations of baseline DSGE model**
 - ❑ **Representative consumer**
 - ❑ **Representative firm**
 - ❑ **Perfect competition in all markets**
 - ❑ **Rational expectations**
 - ❑ **Perfect AD financial markets**
- ❑ **All modern macro models descend from RBC model – dynamic GE**
 - ❑ **No matter how many market imperfections, heterogeneity, etc, etc.**

THE REAL BUSINESS CYCLE MODEL
Kydland and Prescott (1982), Long and Plosser (1983), King, Plosser, and Rebelo (1988)

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- ❑ **All modern macro models descend from RBC model – dynamic GE**
 - ❑ No matter how many market imperfections, heterogeneity, etc, etc.
- ❑ **Foundations of the RBC model**
 - ❑ Without optimizing consumers: Solow growth model
 - ❑ With optimizing consumers: Ramsey/Cass/Koopmans model
- ❑ **“The Solow model”**

TOWARDS DYNAMIC GENERAL EQUILIBRIUM

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- Model of non-asset income so far: endowment y_t , possibly stochastic
- Now suppose y_t is **labor income**

$$y_t = w_t n_t$$

- Normalize “time available” in each time period to one unit
 - Individual decides how to divide between “labor” and “leisure”
 - (Basic models: leisure is all “non-labor,” but empirical and theoretical work recently studying the importance of finer categorizations of “non-labor time” for macro issues – e.g., search and matching theory)
 - **Labor = n_t \leftrightarrow leisure is $l_t = 1-n_t$**

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- Assert that individuals care about leisure, $u(c_t, l_t)$
 - $u_{ct} > 0, u_{lt} > 0, u_{cct} < 0, u_{llt} < 0$
 - Inada conditions on both c and l

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 - ❑ $u_{ct} > 0, u_{lt} > 0, u_{cct} < 0, u_{llt} < 0$
 - ❑ Inada conditions on both c and l
- ❑ Sometimes more convenient to represent as $u(c_t, n_t)$
 - ❑ $u_{ct} > 0, u_{nt} < 0, u_{cct} < 0, u_{nnt} > 0$ (**strictly decreasing and convex in n**)

TOWARDS DYNAMIC GENERAL EQUILIBRIUM

- **Intertemporal optimization problem**

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, n_t^S) \quad \text{subject to} \quad c_t + a_t = w_t n_t^S + (1+r_t)a_{t-1}$$

- **Individual takes as given $\{w_t, r_t\}_{t=0,1,2,\dots}$ -- price-taker in labor market**
 - From perspective of individual, (w, r) evolve as Markov
- **Notation n^S emphasizes individual's **supply** of labor**

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□ Recursive representation

- State vector in arbitrary period t : $[a_{t-1}; w_t, r_t]$

$$V(a_{t-1}; w_t, r_t) = \max_{\{c_t, n_t^S, a_t\}} \left\{ u(c_t, n_t^S) + \beta E_t V(a_t; w_{t+1}, r_{t+1}) \right\}$$

subject to $c_t + a_t = w_t n_t^S + (1+r_t)a_{t-1}$

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□ FOCs

c_t :

n_t^S :

LABOR SUPPLY

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$$\left. \begin{array}{l} \mathbf{c}_t : \quad u_{c_t} - \lambda_t = 0 \\ \mathbf{n}_t^S : \quad u_{n_t} + \lambda_t w_t = 0 \end{array} \right\} \quad -\frac{u_{n_t}(c_t, n_t^S)}{u_{c_t}(c_t, n_t^S)} = w_t \quad \begin{array}{l} \text{CONSUMPTION-LEISURE} \\ \text{OPTIMALITY CONDITION} \\ \text{A static condition} \end{array}$$

LABOR SUPPLY

$$-\frac{u_n(c_t, n_t^S)}{u_c(c_t, n_t^S)} = w_t \quad \Rightarrow \quad n_t^S = n^S(w_t; c_t)$$

- **Consumption-leisure (aka consumption-labor) optimality condition**
 - An **intra-temporal** optimality condition

- **Defines period- t labor supply function**
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 - ...but if representative agent, equivalent to aggregate labor supply
 - Note: for given c

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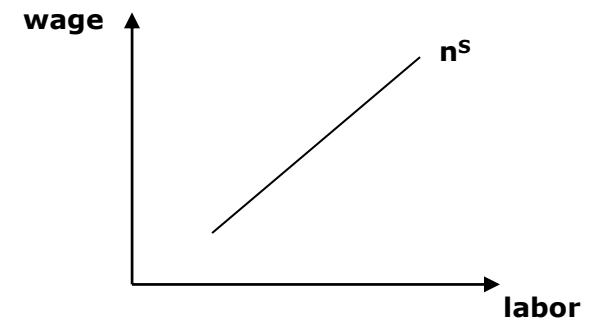
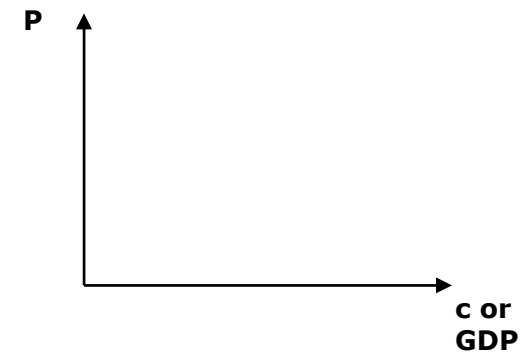
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 - ❑ ...but if representative agent, equivalent to aggregate labor supply
 - ❑ Note: for given c
- ❑ **Example:** $u(c, n) = \ln c - \frac{\theta}{1+1/\psi} n^{1+1/\psi}$
 - ❑ Compute labor supply function?
 - ❑ Compute elasticity of n^S_t with respect to w_t ?
Frisch elasticity of labor supply

THE THREE MACRO (AGGREGATE) MARKETS

☐ **Goods Markets**

☐ **Labor Markets**

☐ **Capital/Savings/Funds/Asset Markets
(aka Financial Markets)**



PRODUCTION OF GOODS

- ❑ **Representative firm** produces the numeraire output good of the economy
- ❑ **A homogenous output good**
- ❑ **Perfect competition in goods supply**
- ❑ **Inputs**
 - ❑ **Labor**
 - ❑ **Capital**
 - ❑ E.g., machines, factories, computers, intangibles, ...
- ❑ **Firm produces using a (aggregate) production technology**

$$y_t = z_t \cdot f(k_t^D, n_t^D)$$
 - ❑ k^D the firm's capital demand
 - ❑ n^D the firm's labor demand
 - ❑ $f(\cdot)$ often assumed CRS (Cobb-Douglas, in particular)
 - ❑ z_t a process that shifts the production function
- ❑ **Empirically identify z_t as Solow residual**
 - ❑ **Growth theory: z deterministic**
 - ❑ **Business cycle theory: z stochastic (Markov)**

PRODUCTION OF GOODS

- ❑ **Representative firm profit maximization**
 - ❑ **Price taker in capital market, labor market, and output market**
 - ❑ **Baseline model(s)**
 - ❑ **Firm hires/rents labor and capital each period**
 - ❑ **Firm does not “own” any capital or labor (without loss of generality if no financial market imperfections)**

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- **FOCs**

$$n_t^D : z_t f_n(k_t^D, n_t^D) - w_t = 0$$

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$n_t^D : z_t f_n(k_t^D, n_t^D) - w_t = 0$ **DEFINES labor demand function $n^D(w_t)$**

$k_t^D : z_t f_k(k_t^D, n_t^D) - r_t^k = 0$ **DEFINES capital demand function $k^D(r_t^k)$**

For a given firm
 If rep. firm, equivalent to aggregate factor demands

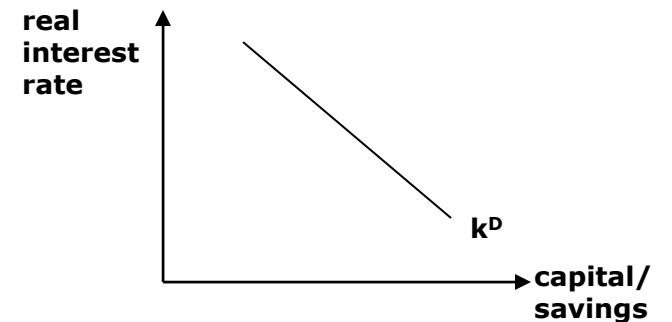
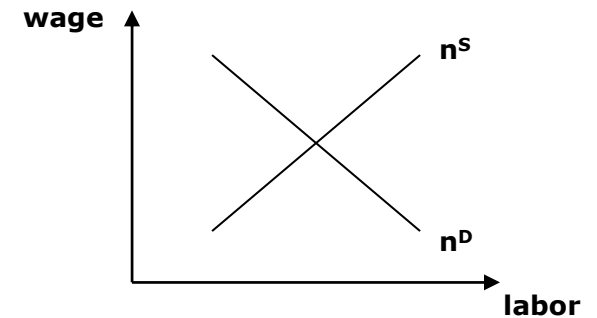
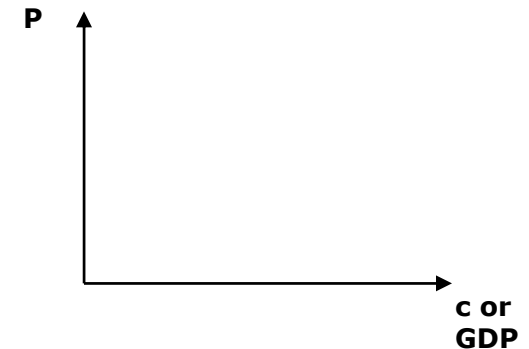
- **Firms entirely static entities in baseline macro model(s)**
 - **Contrast with consumers**
 - **(NK theory and matching theory: firms are dynamic entities)**

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CAPITAL SUPPLY

- **Baseline model(s)**
 - **Physical capital takes “time to build”**
 - **Simplest: one-period lag between building and using capital**
 - **Closed economy**
 - **Aggregate capital demand must be supplied domestically**

- **Consumer intertemporal optimization problem**

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, n_t^S) \quad \text{subject to} \quad c_t + a_t = w_t n_t^S + (1 + r_t) a_{t-1}$$
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 - **Representative agent: a_{t-1} is **economy's** pre-determined stock of assets**

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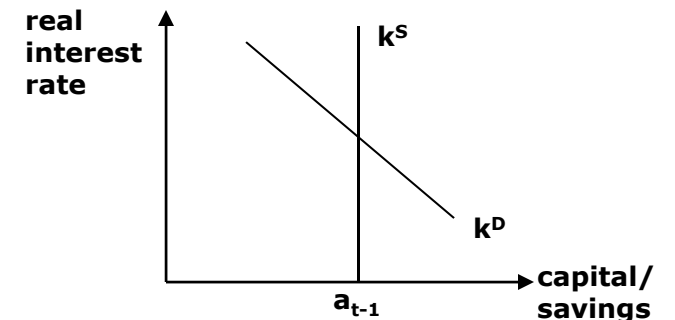
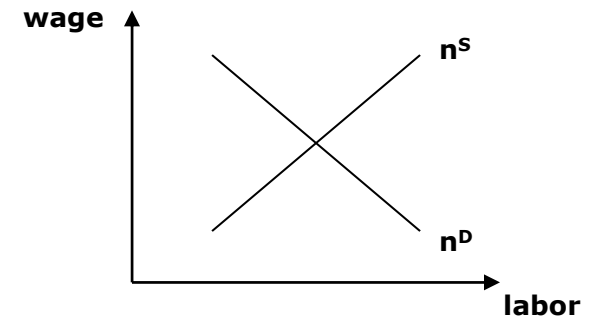
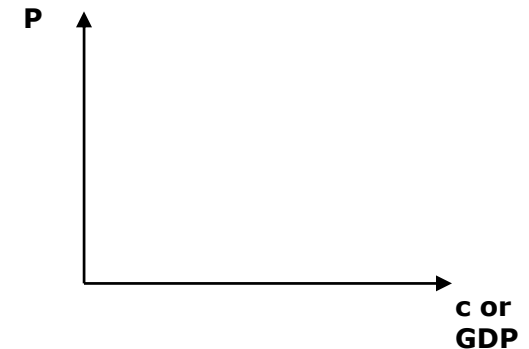
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 - ❑ **Not accounting depreciation**
 - ❑ **Compensation reflected in capital-market-clearing price: $r_t = r^k - \delta$**

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- ❑ Euler equation

$$u'(c_t) = \beta E_t \{ u'(c_{t+1}) (1 + r_{t+1}^k - \delta) \}$$

- ❑ From perspective of single individual: characterizes optimal **savings** (flow!) decision between t and $t+1$
 - ❑ From perspective of entire economy: characterizes optimal **investment** (flow!) in capital stock between t and $t+1$
- ❑ Closed economy: domestic savings = domestic investment
- ❑ Note timing: savings/investment decisions in t alter the available capital stock in period $t+1$ ("time to build")

TOWARDS DYNAMIC GENERAL EQUILIBRIUM

- ❑ Round out final details
- ❑ Baseline model(s)
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 - ❑ i.e., capital good in a given period can be “dismantled” and used for consumption in future periods
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DYNAMIC GENERAL EQUILIBRIUM

- Economy-wide state vector in period t : $(k_t; z_t)$
- Consider $T \rightarrow \text{infinity}$
- **Definition:** a **dynamic stochastic general equilibrium** is time-invariant state-contingent price functions $w(k_t; z_t)$, $r^k(k_t; z_t)$ and state-contingent consumption, labor, and (one-period-ahead) capital decision rules $c(k_t; z_t)$, $n(k_t; z_t)$, and $k(k_t; z_t)$ that **jointly** satisfy the following:

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 3. **(Markets clear)**
 - Labor-market clearing
 $n(k_t; z_t)$ defined as $n^D_t = n^S_t$, for all t
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 - Goods market clearing
 $c(k_t; z_t) + k(k_t; z_t) - (1-\delta)k_t = z_t f(k_t, n(k_t; z_t))$, for all t

given the initial capital stock k_0 and (Markov) transition process for $z_t \rightarrow z_{t+1}$

INTERTEMPORAL MODELS: BASICS OF DYNAMIC PROGRAMMING

MARCH 2, 2017

DYNAMIC PROGRAMMING

- ❑ Can we represent intertemporal problems **recursively?**
 - ❑ Rather than **sequentially**

- ❑ **Benefits**
 - ❑ Allows application of series of theorems/results that guarantees (conditional on model...) a **solution exists in the space of functions**
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 - ❑ Sometimes (often?) not obvious how to recast sequential problem as recursive problem

- ❑ **Ljungqvist and Sargent (2012, Preface p. 34)**

*“The art in applying recursive methods is to find a convenient definition of a state. It is often not obvious what the state is, or even whether a **finite-dimensional** state exists.”*

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- ❑ **Start with deterministic case**
 - ❑ **(Fairly) straightforward**
 - ❑ **Stochastic case requires more structure**

FROM SEQUENTIAL TO RECURSIVE

- Lagrangian of consumer problem, with planning horizon T

$$\max_{\{c_t, a_t\}_{t=0}^{\infty}} \sum_{t=0}^T \beta^t \left[u(c_t) + \lambda_t (y_t + (1 + r_{t-1})a_{t-1} - c_t - a_t) \right]$$

- **State variables** of consumer problem at beginning of any period s
 - a_{s-1} (accumulation variable)
 - r_s (price-taker)
 - A sufficient summary of the **dynamic position of the environment** in which the consumer operates

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$$V^0(a_{-1}, r_0; \cdot) \equiv \max_{\{c_t, a_t\}_{t=0}^{\infty}} \sum_{t=0}^T \beta^t \left[u(c_t) + \lambda_t (y_t + (1 + r_{t-1})a_{t-1} - c_t - a_t) \right]$$

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 - r_s (price-taker)
 - A sufficient summary of the **dynamic position of the environment** in which the consumer operates
- Define $V^0(a_{-1}, r_0; \cdot)$ as value function starting from period zero
 - The maximized **value** of the constrained optimization problem
 - As function of period-zero parameters of the problem
- **Goal:** recast problem of finding optimal **sequence** $\{c_t, a_t\}_{t=0,1,2,\dots,T}$ into problem of finding **functions** $\{V^i(\cdot)\}_{t=0,1,2,\dots,T}$
 - (Actually, find $V^i(\cdot)$ along with two other functions)

FROM SEQUENTIAL TO RECURSIVE

- Write out more explicitly

$$V^0(a_{-1}, r_0; \cdot) \equiv \max_{\{c_0, a_0, c_t, a_t\}_{t=1}^{\infty}} \left\{ \begin{aligned} &u(c_0) + \lambda_0 (y_0 + (1 + r_{-1})a_{-1} - c_0 - a_0) \\ &+ \sum_{t=1}^T \beta^t \left[u(c_t) + \lambda_t (y_t + (1 + r_{t-1})a_{t-1} - c_t - a_t) \right] \end{aligned} \right\}$$

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↓ Separate terms

$$V^0(a_{-1}, r_0; \cdot) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 (y_0 + (1 + r_{-1})a_{-1} - c_0 - a_0) \right\} \\ + \max_{c_0, a_0} \left\{ \max_{\{c_t, a_t\}_{t=1}^{\infty}} \sum_{t=1}^T \beta^t [u(c_t) + \lambda_t (y_t + (1 + r_{t-1})a_{t-1} - c_t - a_t)] \right\}$$

Note the max inside the max

↓ Adjust β factors

FROM SEQUENTIAL TO RECURSIVE

Adjust β
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→

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$V^0(a_{-1}, r_0; \cdot)$ is value function starting from period 0.
Bellman Principle of Optimality:
 optimal decisions in the initial period induce a future state, from which (future) decisions are optimal (Bellman, 1957)

$= V^1(a_0, r_1; \cdot)$, value function starting from period 1.
 The value resulting from optimal decisions starting from period 1.

FROM SEQUENTIAL TO RECURSIVE

Adjust β
factors
→

$$V^0(a_{-1}, r_0; \cdot) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 (y_0 + (1 + r_{-1})a_{-1} - c_0 - a_0) \right\}$$

$$+ \beta \cdot \max_{c_0, a_0} \left\{ \max_{\{c_t, a_t\}_{t=1}^{\infty}} \sum_{t=1}^T \beta^{t-1} \left[u(c_t) + \lambda_t (y_t + (1 + r_{t-1})a_{t-1} - c_t - a_t) \right] \right\}$$

$V^0(a_{-1}, r_0; \cdot)$ is value function starting from period 0.
Bellman Principle of Optimality: optimal decisions in the initial period induce a future state, from which (future) decisions are optimal (Bellman, 1957)

$= V^1(a_0, r_1; \cdot)$, value function starting from period 1.
 The value resulting from optimal decisions starting from period 1.

Recursive representation of consumer problem

$$V^0(a_{-1}, r_0; \cdot) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 (y_0 + (1 + r_{-1})a_{-1} - c_0 - a_0) + \beta \cdot V^1(a_0, r_1; \cdot) \right\}$$

- ❑ **Bellman Equation**
- ❑ **Can analyze optimization problem for period zero...**
 - ❑ ...given **Bellman Principle of Optimality** holds
 - ❑ **(But how do $V^0(\cdot)$ and $V^1(\cdot)$ relate to each other?)**

BELLMAN EQUATION

□ Bellman Equation

$$V^0(a_{-1}, r_0; \cdot) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 (y_0 + (1 + r_{-1})a_{-1} - c_0 - a_0) + \beta \cdot V^1(a_0, r_1; \cdot) \right\}$$

- Starting point for recursive analysis
- Applicable to finite T -period or $T \rightarrow \infty$ problems
- Construction requires identifying **state variables** of optimization problem

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 - Applicable to finite T -period or $T \rightarrow \infty$ problems
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 - **T -period problem**
 - Solution involves sequence of functions $V^0(\cdot), V^1(\cdot), \dots, V^{T-1}(\cdot), V^T(\cdot)$
 - $V^i(\cdot)$ functions in general will differ – reflecting time until end of planning horizon
 - E.g., maximized value starting from age = 60 different from maximized value starting from age = 30 (intuitively)

 - **Infinite-horizon problem**
 - **Deterministic case:** $V(\cdot) = V^i(\cdot) = V^j(\cdot)$ all i, j
 - Always an infinity of periods left to go
- Stochastic case?**
Requires more structure...

BELLMAN EQUATION

□ **Bellman Equation (for $T \rightarrow \infty$)**

$$V(a_{-1}, r_0; \cdot) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 (y_0 + (1 + r_{-1})a_{-1} - c_0 - a_0) + \beta \cdot V(a_0, r_1; \cdot) \right\}$$

□ **Use to characterize optimal decisions**

□ **Period-0 FOCs**

$$c_0: \quad u'(c_0) - \lambda_0 = 0$$

$$a_0: \quad -\lambda_0 + \beta V_1(a_0, r_1; \cdot) = 0$$

How to compute $V_1(\cdot)$?

BELLMAN EQUATION

□ **Bellman Equation (for $T \rightarrow \infty$)**

$$V(a_{-1}, r_0; \cdot) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 (y_0 + (1 + r_{-1})a_{-1} - c_0 - a_0) + \beta \cdot V(a_0, r_1; \cdot) \right\}$$

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$$c_0: \quad u'(c_0) - \lambda_0 = 0$$

$$a_0: \quad -\lambda_0 + \beta V_1(a_0, r_1; \cdot) = 0 \qquad \text{How to compute } V_1(\cdot)?$$

Return to this ... □ **Suppose** optimal choice characterized by $c_0 = c(a_{-1}; \cdot)$, $a_0 = a(a_{-1}; \cdot)$ ($c(\cdot)$ and $a(\cdot)$ **time-invariant functions** in infinite-period problem)

□ **Insert in value function (can now drop max operator)**

$$V(a_{-1}, r_0; \cdot) \equiv u(c(a_{-1})) + \lambda_0 (y_0 + (1 + r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1})) + \beta \cdot V(a(a_{-1}), r_1; \cdot)$$

□ **Now compute marginal**

BELLMAN EQUATION

□ **Bellman Equation (for $T \rightarrow \infty$)**

$$V(a_{-1}, r_0; \cdot) \equiv u(c(a_{-1})) + \lambda_0 (y_0 + (1 + r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1})) + \beta \cdot V(a(a_{-1}), r_1; \cdot)$$

□ **Now compute marginal (suppress r argument of $c(\cdot)$ and $a(\cdot)$ functions)**

$$V_1(a_{-1}, r_0; \cdot) =$$

BELLMAN EQUATION

□ **Bellman Equation (for $T \rightarrow \infty$)**

$$V(a_{-1}, r_0; \cdot) \equiv u(c(a_{-1})) + \lambda_0 (y_0 + (1 + r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1})) + \beta \cdot V(a(a_{-1}), r_1; \cdot)$$

□ **Now compute marginal (suppress r argument of $c(\cdot)$ and $a(\cdot)$ functions)**

$$V_1(a_{-1}, r_0; \cdot) = u'(c_0) \cdot c'(a_{-1}) + \lambda_0(1 + r_{-1}) - \lambda_0 \cdot c'(a_{-1}) - \lambda_0 \cdot a'(a_{-1}) + \beta V_1(a_0, r_1; \cdot) \cdot a'(a_{-1})$$

BELLMAN EQUATION

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□ **Now compute marginal (suppress r argument of $c(\cdot)$ and $a(\cdot)$ functions)**

$$\begin{aligned} V_1(a_{-1}, r_0; \cdot) &= u'(c_0) \cdot c'(a_{-1}) + \lambda_0(1 + r_{-1}) - \lambda_0 \cdot c'(a_{-1}) - \lambda_0 \cdot a'(a_{-1}) + \beta V_1(a_0, r_1; \cdot) \cdot a'(a_{-1}) \\ &= \lambda_0(1 + r_{-1}) + \underbrace{[u'(c_0) - \lambda_0] \cdot c'(a_{-1})}_{= 0 \text{ by period-0 FOCs}} + \underbrace{[-\lambda_0 + \beta V_1(a_0, r_1; \cdot)] \cdot a'(a_{-1})}_{= 0 \text{ by period-0 FOCs}} \end{aligned}$$

BELLMAN EQUATION

□ **Bellman Equation (for $T \rightarrow \infty$)**

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□ **Now compute marginal (suppress r argument of $c(\cdot)$ and $a(\cdot)$ functions)**

$$\begin{aligned} V_1(a_{-1}, r_0; \cdot) &= u'(c_0) \cdot c'(a_{-1}) + \lambda_0(1 + r_{-1}) - \lambda_0 \cdot c'(a_{-1}) - \lambda_0 \cdot a'(a_{-1}) + \beta V_1(a_0, r_1; \cdot) \cdot a'(a_{-1}) \\ &= \lambda_0(1 + r_{-1}) + \underbrace{[u'(c_0) - \lambda_0] \cdot c'(a_{-1})}_{= 0 \text{ by period-0 FOCs}} + \underbrace{[-\lambda_0 + \beta V_1(a_0, r_1; \cdot)] \cdot a'(a_{-1})}_{= 0 \text{ by period-0 FOCs}} \end{aligned}$$

$$\Rightarrow V_1(a_{-1}, r_0; \cdot) = \lambda_0(1 + r_{-1})$$

Envelope Condition

□ **Envelope Theorem**

Note: **envelope theorem** has nothing to do with dynamic programming

- **In computing first-order effects of changes in a problem's parameters on the maximized value, can ignore how optimal choices will adjust**
 - **Intuition: because already at a max (marginal costs = marginal benefits)**
- **Need only consider the direct effect**

BELLMAN EQUATION

□ **Bellman Equation (for $T \rightarrow \infty$)**

$$V(a_{-1}, r_0; \cdot) \equiv u(c(a_{-1})) + \lambda_0 (y_0 + (1 + r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1})) + \beta \cdot V(a(a_{-1}), r_1; \cdot)$$

□ **Now compute marginal (suppress r argument of $c(\cdot)$ and $a(\cdot)$ functions)**

$$\begin{aligned} V_1(a_{-1}, r_0; \cdot) &= u'(c_0) \cdot c'(a_{-1}) + \lambda_0(1 + r_{-1}) - \lambda_0 \cdot c'(a_{-1}) - \lambda_0 \cdot a'(a_{-1}) + \beta V_1(a_0, r_1; \cdot) \cdot a'(a_{-1}) \\ &= \lambda_0(1 + r_{-1}) + \underbrace{[u'(c_0) - \lambda_0] \cdot c'(a_{-1})}_{= 0 \text{ by period-0 FOCs}} + \underbrace{[-\lambda_0 + \beta V_1(a_0, r_1; \cdot)] \cdot a'(a_{-1})}_{= 0 \text{ by period-0 FOCs}} \end{aligned}$$

$$\Rightarrow V_1(a_{-1}, r_0; \cdot) = \lambda_0(1 + r_{-1}) \xrightarrow{\text{evaluate at period 1}} V_1(a_0, r_1; \cdot) = \lambda_1(1 + r_0) \quad \text{Envelope Condition}$$

□ **Envelope Theorem**

Note: **envelope theorem** has nothing to do with dynamic programming

- **In computing first-order effects of changes in a problem's parameters on the maximized value, can ignore how optimal choices will adjust**
 - **Intuition: because already at a max (marginal costs = marginal benefits)**
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BELLMAN EQUATION

□ **Bellman Equation (for $T \rightarrow \infty$)**

$$V(a_{-1}, r_0; \cdot) \equiv u(c(a_{-1})) + \lambda_0 (y_0 + (1 + r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1})) + \beta \cdot V(a(a_{-1}), r_1; \cdot)$$

□ **Use to characterize optimal decisions**

□ **Period-0 FOCs, now evaluated using $c(a_{-1}), a(a_{-1})$**

$$c_0: \quad u'(c(a_{-1})) - \lambda_0 = 0$$

$$a_0: \quad -\lambda_0 + \beta V_1(a_0(a_{-1}), r_1; \cdot) = 0$$

$$\text{Env: } V_1(a(a_{-1}), r_1; \cdot) = \lambda_1 (1 + r_0)$$

BELLMAN EQUATION

□ **Bellman Equation (for $T \rightarrow \infty$)**

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□ **Seems like usual Euler equation from sequential analysis (deterministic)...**

DETERMINISTIC – RECURSIVE ANALYSIS

- Solution of infinite-horizon consumer problem (starting from date zero)...
- ...is a consumption **decision rule** $c(a_{-1}; \cdot)$, asset **decision rule** $a(a_{-1}; \cdot)$, and **value function** $V(a_{-1}; \cdot)$ that satisfies

- **Bellman equation**

$$V(a_{-1}, r_0; \cdot) \equiv u(c(a_{-1})) + \lambda_0 (y_0 + (1 + r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1})) + \beta \cdot V(a(a_{-1}), r_1; \cdot)$$

- **Euler equation**

by envelope theorem

$$u'(c(a_{-1})) = \beta V_1(a(a_{-1}), r_1; \cdot) \quad \longleftrightarrow \quad u'(c(a_{-1})) = \beta(1 + r_0)u'(c(a_0))$$

- which is the TVC in the limit $t \rightarrow \infty$: $\lim_{t \rightarrow \infty} \beta^t u'(c(a_{t-1}^*)) \cdot a(a_{t-1}^*) = 0$

- **Budget constraint**

$$y_0 + (1 + r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) = 0$$

taking as given (a_{-1}, r_0, r_{-1})

DETERMINISTIC – SEQUENTIAL ANALYSIS

- Solution of infinite-horizon consumer problem (starting from date zero)...
- is a consumption and asset **sequence** $\{c_t^*, a_t^*\}_{t=0}^{\infty}$ that satisfies

- **Sequence of flow budget constraints**

$$c_t^* + a_t^* = y_t + (1 + r_{t-1})a_{t-1}^*, \quad t = 0, 1, 2, \dots$$

- **Sequence of Euler equations**

$$u'(c_t^*) = \beta u'(c_{t+1}^*)(1 + r_t), \quad t = 0, 1, 2, \dots$$

- which is the TVC in the limit $t \rightarrow \infty$: $\lim_{t \rightarrow \infty} \beta^t u'(c_t^*) a_t^* = 0$

taking as given $\left(\{r_t, y_t\}_{t=0}^{\infty}, a_{-1}, r_{-1} \right)$

Does solution to recursive problem coincide with solution to sequential problem?

RECURSIVE VS. SEQUENTIAL ANALYSIS

- ❑ **Does solution to recursive problem coincide with solution to sequential problem?**
- ❑ **Does solution to sequential problem coincide with solution to recursive problem?**
- ❑ **In general, no!**
- ❑ **No reason why it should!**

RECURSIVE VS. SEQUENTIAL ANALYSIS

- ❑ Does solution to recursive problem coincide with solution to sequential problem?
- ❑ Does solution to sequential problem coincide with solution to recursive problem?
- ❑ In general, no!
- ❑ No reason why it should!
- ❑ In constructing Bellman representation ($T \rightarrow \infty$ case), the **imposition of time-invariant functions $c(a)$, $a(a)$ potentially limits the class of solutions**
 - ❑ In original sequential formulation, this is neither explicitly nor implicitly a requirement of the solution!
- ❑ In general (here without proof...)
 - ❑ Solution to the sequential problem is also a solution to the recursive problem
 - ❑ Solution to the recursive problem is also a solution to the sequential problem **provided some further regularity conditions hold**
- ❑ Stokey, Lucas, Prescott text (1989)

RECURSIVE VS. SEQUENTIAL ANALYSIS

- So why go recursive?

- Allows application of series of theorems/results that guarantee a **solution exists in the space of functions**
- Allows application of series of theorems/results that help **find solution in the space of functions**

Underlying theory:

Contraction Mapping Theorem, Blackwell's Sufficient Conditions for a Contraction, Theorem of the Maximum

THEORY

□ **Blackwell's Sufficient Conditions for a Contraction:** Let X be a subset of R' and let $B(X)$ be the set of bounded functions $f : X \rightarrow R$ with the sup norm. Let $T : B(X) \rightarrow B(X)$ be an operator satisfying

a. (Monotonicity) $f, g \in B(X)$ and $f(x) \leq g(x)$, for all $x \in X$, implies $(Tf)(x) \leq (Tg)(x)$, for all $x \in X$

b. (Discounting) There exists some $\beta \in (0,1)$ such that

$$[T(f+a)](x) \leq (Tf)(x) + \beta a, \text{ for all } f \in B(X), a \geq 0, x \in X$$

Then T is a contraction with modulus β .

(Note: $(f+a)(x)$ is the function defined by $(f+a)(x) = f(x) + a$)

THEORY

- Let (S, ρ) be a metric space and $T : S \rightarrow S$ be a function mapping set S into itself. T is a **contraction mapping (with modulus β)** if for some $\beta \in (0,1)$, $\rho(Tx, Ty) \leq \beta\rho(x, y)$ for all $x, y \in S$.
Example: $S = [a, b]$ with $\rho(x, y) = |x - y|$ (Euclidean norm)

- **Contraction Mapping Theorem:** If (S, ρ) is a metric space and $T : S \rightarrow S$ is a contraction mapping with modulus β , then
 - a. T has exactly one fixed point v in set S .
 - b. For any $v_0 \in S$, $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$ for $n = 0, 1, 2, \dots$

- CMT states that a contraction mapping has a **unique fixed point**, and the fixed point can be found by iterative application of the mapping T starting starting from any point in S .

THEORY

- **General class of problems to which our (usual) economic optimization problems belong have the form**

$$(Tv)(x) = \sup_{y \in r(x)} [F(x,y) + \beta v(y)]$$

- **For our economic theory: would like operator T to map the space $C(X)$ of bounded continuous functions of the state vector into itself. Would also like to be able to characterize the set of maximizing values of y given x .**

THEORY

- **General class of problems to which our (usual) economic optimization problems belong have the form**

$$(Tv)(x) = \sup_{y \in \Gamma(x)} [F(x,y) + \beta v(y)]$$
- **For our economic theory: would like operator T to map the space $C(X)$ of bounded continuous functions of the state vector into itself. Would also like to be able to characterize the set of maximizing values of y given x .**
- **Theorem of the Maximum:** Let X be a subset of R^l , Y be a subset of R^m , let $f : X \times Y \rightarrow R$ be a (single-valued) continuous function, and let $\Gamma : X \rightarrow Y$ be a compact-valued and continuous correspondence. The problem we are interested in is of the form $\sup_{y \in \Gamma(x)} f(x,y)$. Then
 - a. \sup can be replaced with \max because, for each x , the maximum is attained and the function $h(x) = \max_{y \in \Gamma(x)} f(x,y)$ is well defined and continuous
 - b. The correspondence $G(x) = \{y \in \Gamma(x) : f(x,y) = h(x)\}$ is well defined, is non-empty, is compact-valued, and upper hemi-continuous.
- **Theorem of the Maximum establishes the **existence** of the maximum of the problem.**

THEORY

- Suppose in addition to the hypotheses of the Theorem of the Maximum, the correspondence Γ is convex-valued and the function f is strictly concave in y .

→ Then G is single-valued. Call this function g , and g is continuous.

- Establishes that, given these conditions and given the unique solution of the Bellman Equation, there is **a unique g that is the optimal “decision rule.”**

- If $\{f_n(x, y)\}$ is a sequence of continuous functions converging to $f(x, y)$, each strictly concave in y , then the sequence of functions $\{g_n(x)\}$ (which are the argmax of the sequence $\{f_n(x, y)\}$) converges pointwise to $g(x)$, which is the argmax of $f(x, y)$.

- The latter result is very useful considered in the context of the Contraction Mapping Theorem. **It guarantees that the solutions to the sequence of problems converges to the true solution.**

RECURSIVE VS. SEQUENTIAL ANALYSIS

- ❑ So why go recursive?

Underlying theory:

Contraction Mapping Theorem, Blackwell's Sufficient Conditions for a Contraction, Theorem of the Maximum

- ❑ Allows application of series of theorems/results that guarantee a **solution exists in the space of functions**
- ❑ Allows application of series of theorems/results that help **find solution in the space of functions**

- ❑ Computational algorithms require it – computers can't handle infinite-dimensional objects!

RECURSIVE VS. SEQUENTIAL ANALYSIS

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Underlying
theory:

Contraction Mapping Theorem, Blackwell's Sufficient Conditions for a Contraction, Theorem of the Maximum

- ❑ Allows application of series of theorems/results that guarantee a **solution exists in the space of functions**
 - ❑ Allows application of series of theorems/results that help **find solution in the space of functions**
- ❑ Computational algorithms require it – computers can't handle infinite-dimensional objects!
- ❑ Can't really "choose" whether want to analyze problem sequentially or recursively
 - ❑ All but the most limited of problems/models require computational solution
 - ❑ In which case model analysis **is** recursive
- ❑ What about **stochastic** dynamic programming?
 - ❑ Even more structure required....

STOCHASTIC DYNAMIC PROGRAMMING

- ❑ **Even more structure required on the problem to recursively solve dynamic stochastic optimization problems**

- ❑ **Main (new) technical problem**
 - ❑ **Branching** of event tree at each of T periods (possibly $T \rightarrow \infty$)

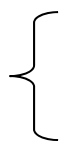
- ❑ **Main technical solution/assumption**
 - ❑ **Assume risk follows Markov process**
 - ❑ **Which enables series of theorems/results from deterministic dynamic programming to work in stochastic case...**
 - ❑ **...given further technical regularity assumptions**

RECURSIVE REPRESENTATION

□ State variables

- A sufficient summary, as of the start of period t , of the dynamic position of the environment in which the maximizing agent operates
- “Environment” of the agent – what needs to be known in order to optimize in period t ?

The usual suspects



- Individual-specific quantities
- Market prices
- Government policies
- (Fixed structural parameters – will omit from state vector for parsimony)



Important: states can be endogenous or exogenous

□ Ljungqvist and Sargent (2012, Preface p. 34)

*“The art in applying recursive methods is to find a convenient definition of a state. It is often not obvious what the state is, or even whether a **finite-dimensional state** exists.”*

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The usual suspects

- Individual-specific quantities
 - Market prices
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Important: states can be endogenous or exogenous

- “Sufficient” – there are no other objects (quantities, prices, govt policies, etc.) that must be known in order to optimize in period t

- Concept well-defined for both finite- T and $T \rightarrow \infty$ problems

- **KEY: Period- t decisions are function of the period- t state variables**

□ Ljungqvist and Sargent (2012, Preface p. 34)

*“The art in applying recursive methods is to find a convenient definition of a state. It is often not obvious what the state is, or even whether a **finite-dimensional state** exists.”*

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- Starting point for recursive analysis
- Applicable to finite T -period or $T \rightarrow \infty$ problems
- Construction requires identifying **state variables**

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 - Solution involves sequence of functions $V^0(\cdot), V^1(\cdot), \dots, V^{T-1}(\cdot), V^T(\cdot)$
 - $V^i(\cdot)$ functions in general will differ – reflecting time until end of planning horizon
 - E.g., maximized value starting from age = 60 different from maximized value starting from age = 30 (intuitively)

- **Infinite-horizon problem (“stationary” environment)**
 - **Deterministic case: $V(\cdot) = V^i(\cdot) = V^j(\cdot)$ all i, j**
 - Always an infinity of periods left to go

BELLMAN EQUATION

□ **Bellman Equation (for $T \rightarrow \infty$)**

$$V(a_{-1}, r_0; \cdot) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 (y_0 + (1 + r_{-1})a_{-1} - c_0 - a_0) + \beta \cdot V(a_0, r_1; \cdot) \right\}$$

□ **Use to characterize optimal decisions**

□ **Period-0 FOCs, evaluated using time-invariant $c(a_{-1}), a(a_{-1})$**

$$\left. \begin{aligned} c_0: \quad & u'(c(a_{-1})) - \lambda_0 = 0 \\ a_0: \quad & -\lambda_0 + \beta V_1(a_0(a_{-1}), r_1; \cdot) = 0 \\ \text{Env:} \quad & V_1(a(a_{-1}), r_1; \cdot) = \lambda_1(1 + r_0) \end{aligned} \right\} u'(c(a_{-1})) = \beta(1 + r_0)u'(c(a_0))$$

□ **Seems like usual Euler equation from sequential analysis (deterministic)...**

BELLMAN EQUATION

□ Bellman Equation (for $T \rightarrow \infty$)

$$V(a_{-1}, r_0; \cdot) \equiv u(c(a_{-1})) + \lambda_0 (y_0 + (1 + r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1})) + \beta \cdot V(a(a_{-1}), r_1; \cdot)$$

□ Seems like a two-period problem

- In terms of (value) functions, not in terms of choice variables
- Optimize in current period
- Optimize next period (Bellman's Principle of Optimality)

BELLMAN EQUATION

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□ **Seems like a two-period problem**

- **In terms of (value) functions, not in terms of choice variables**
- **Optimize in current period**
- **Optimize next period (Bellman's Principle of Optimality)**

□ **Common notation**

- **Use x for current-period variables**
- **Use x' for next-period variables**

□ **Bellman Equation**

$$V(a, r; \cdot) \equiv u(\underbrace{c(a)}_{= c}) + \lambda (y + (1 + r_{-1})a - \underbrace{c(a)}_{= c} - \underbrace{a(a)}_{= a'}) + \beta \cdot V(\underbrace{a(a)}_{= a'}, r'; \cdot)$$

□ **Euler equation**

$$u'(\underbrace{c(a)}_{= c}) = \beta(1 + r)u'(\underbrace{c(a')}_{= c'})$$

RECURSIVE VS. SEQUENTIAL ANALYSIS

❑ So why go recursive?

- ❑ Allows application of series of theorems/results that guarantee a **solution exists in the space of functions**
- ❑ Allows application of series of theorems/results that help **find solution in the space of functions**

Underlying
Theory:

Contraction Mapping Theorem, Blackwell's Sufficient Conditions for a Contraction, Theorem of the Maximum

- ❑ Suppose $V(\cdot)$ exists
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↓ initial guess (some parametric form)

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If no, insert $V^2(\cdot)$ on RHS and repeat

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e.g., value
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- ❑ Can't "choose" whether to analyze problem sequentially or recursively
 - ❑ All but the most limited of problems require computational solution
 - ❑ In which case model analysis **is** recursive
- ❑ "Solving model sequentially"
 - ❑ Doesn't seem recursive... \longrightarrow $u(c_t) = \beta(1+r_t)u'(c_{t+1})$

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❑ "Solving model sequentially"

- ❑ Doesn't seem recursive...
- ❑ ...but computational implementation **requires** time-invariant decision rule

$$\begin{array}{l}
 \longrightarrow u(c_t) = \beta(1+r_t)u'(c_{t+1}) \\
 \downarrow \text{Imposing recursivity on solution} \\
 u(c(a_{t-1})) = \beta(1+r_t)u'(c(a_t))
 \end{array}$$

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 - ❑ In which case model analysis **is** recursive
- ❑ What about **stochastic** dynamic programming?
 - ❑ Even more structure required....
 - ❑ The key assumption is **Markov risk**

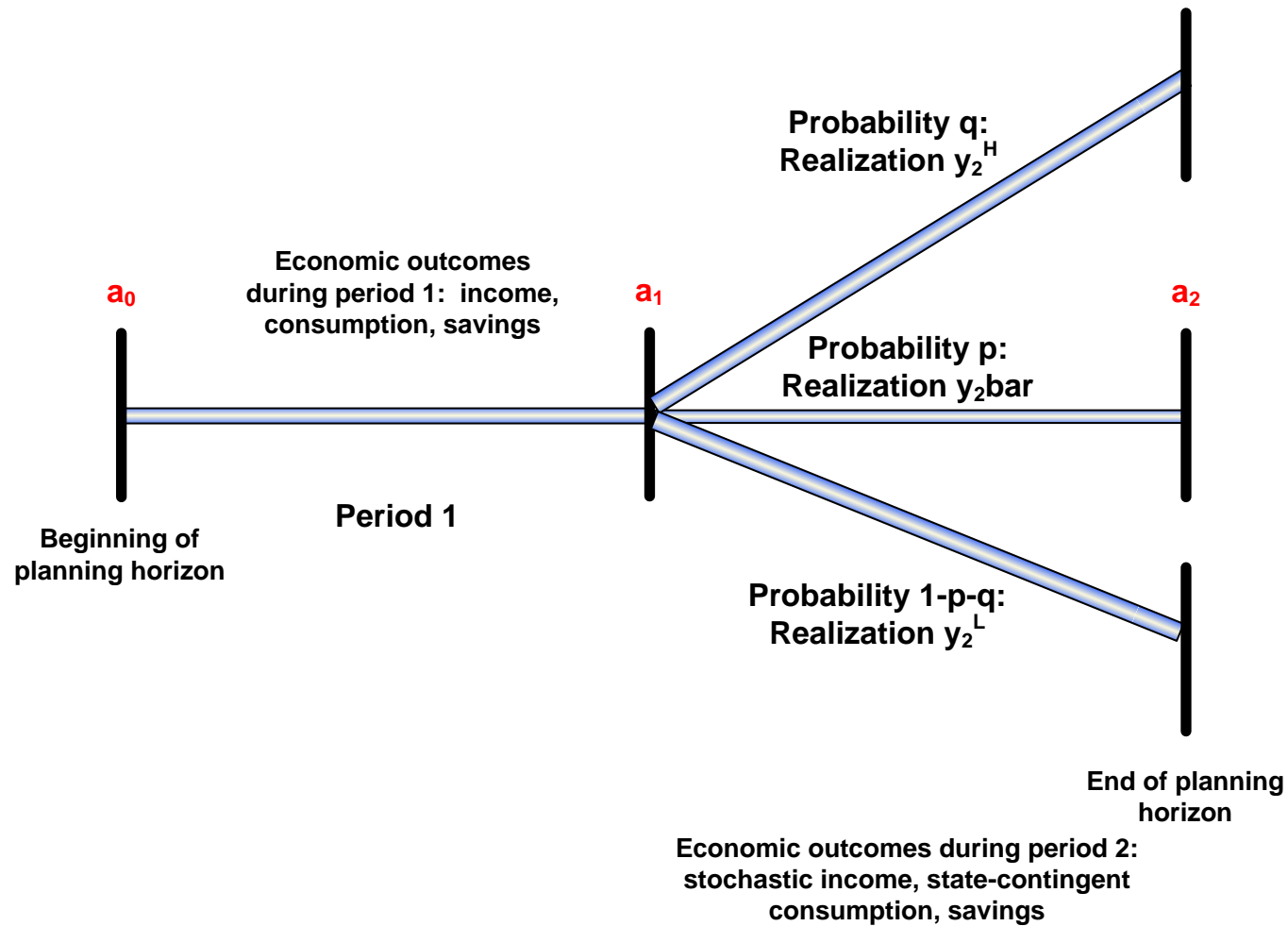
STOCHASTIC DYNAMIC PROGRAMMING

- ❑ Even more structure required on the problem to recursively solve dynamic stochastic optimization problems

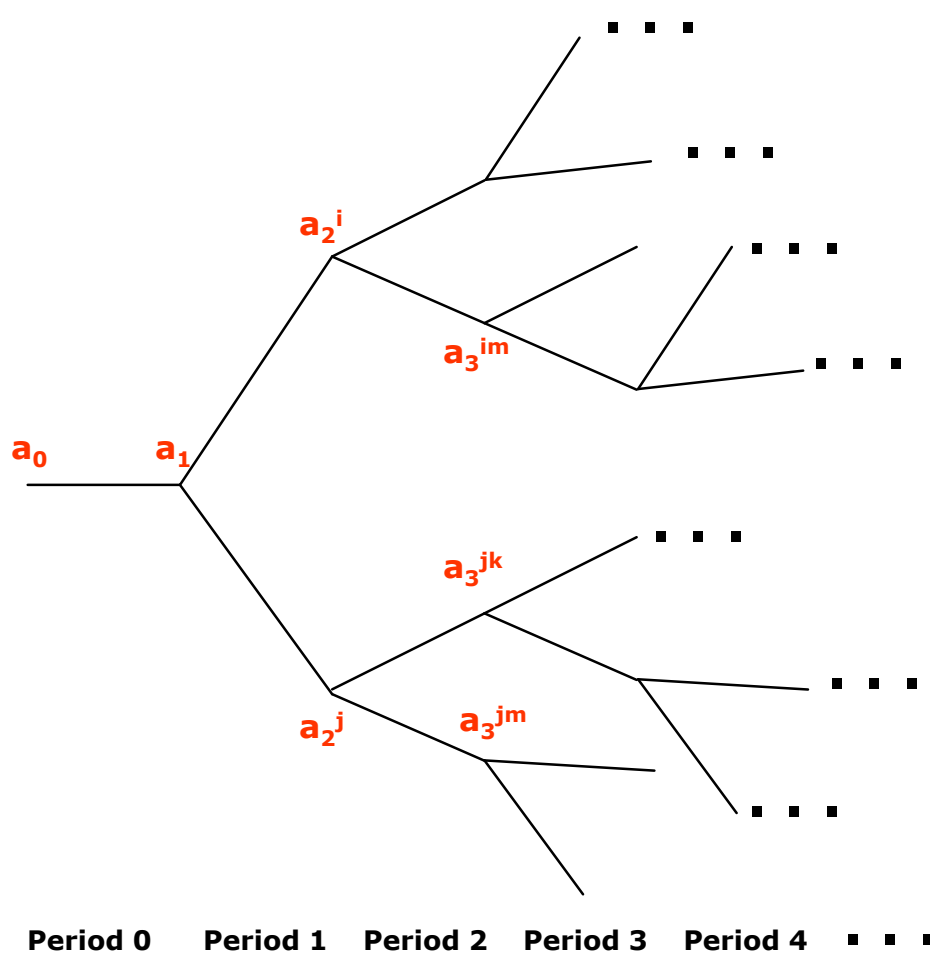
- ❑ **Main (new) technical problem**
 - ❑ **Branching** of event tree at each of T periods (possibly $T \rightarrow \infty$)

- ❑ **Main technical solution/assumption**
 - ❑ Assume risk follows **Markov** process
 - ❑ Which enables series of theorems/results from deterministic dynamic programming to work in stochastic case...
 - ❑ ...given further technical regularity assumptions

EVENT TREE



EVENT TREE



- In general could be any arbitrary unfolding of exogenous risk
e.g., if state i realized in period 16, then 14 possible states in period 17; but if state j realized in period 16, then 8 possible states in period 17

OR

- e.g., probability of state i in period t depends on event in period $t-100000$
- Number of decision rules to solve explodes
- **Intractable!!!**
- **"Curse of dimensionality"**
- **Requires a lot of structure on exogenous risk**

RISK STRUCTURE

Assumptions

- Set of realizations of exogenous state variable is independent of date

$$S_2 = S_3 = S_4 = S_5 = \dots = S_{T-1} = S_T$$

RISK STRUCTURE

Assumptions

- Set of realizations of exogenous state variable is independent of date
- Probability of realization of exogenous state variable in period t depends only on outcomes in period $t-1$
 - Suppose X_t is a stochastic process and x_t is a particular realization
 - X_t is a Markov process if

$$\Pr(X_t = x_t \mid X_{t-1} = x_{t-1}, X_{t-2} = x_{t-2}, X_{t-3} = x_{t-3}, \dots, X_{t-10000} = x_{t-10000}, \dots)$$

$$= \Pr(X_t = x_t \mid X_{t-1} = x_{t-1}) \quad \text{CONDITIONAL probability depends on only } t-1$$

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- ❑ Not as restrictive as it may seem – could have finite lags in process

- ❑ E.g.

- ❑ Just can't have **infinite** lags (in principle) or **"too many"** (finite) lags (in computational practice)

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- ❑ Not as restrictive as it may seem – could have finite lags in process
- ❑ Exogenous state variable is Markov process + assumption/result that decision rules are time-invariant (for $T \rightarrow \infty$) functions of state variables

\Rightarrow **Endogenous processes are Markov given several regularity assumptions**

Underlying theory:
Stokey, Lucas, Prescott
(1989, Chapters 8-12)

STOCHASTIC – SEQUENTIAL ANALYSIS

- ❑ **Planning horizon $T \rightarrow \infty$**
- ❑ **Exogenous state drawn from set S (could be continuous or discrete)**
- ❑ **Suppose single asset with state-contingent r (will illustrate main ideas)**

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- **FOCs**

c_0 :

a_0 :

c_1 :

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 Holds for each
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→
Just as in
stochastic two-
period model

$$1 = E_0 \left[\frac{\beta \lambda_1}{\lambda_0} (1+r_1) \right]$$

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- $\{X_t\}_{t=0,1,2,\dots}$ is Markov process (exogenous and endogenous states)
 - **Nothing about the probability distribution of X_{t+2} is known in period t that is not known in period $t+1$**
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- Allows applying a law of iterated expectations
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$$E_0 \lambda_1 = \beta E_0 [\lambda_2 (1 + r_2)] \longrightarrow E_0 \lambda_1 = \beta E_0 [E_1 (\lambda_2 (1 + r_2))]$$

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$$\mathbf{c}_1: \quad \beta E_0 u'(c_1) - \beta E_0 \lambda_1 = 0$$

→
Holds for each state

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$$\mathbf{a}_1: \quad -\beta E_0 \lambda_1 + \beta^2 E_0 [\lambda_2 (1+r_2)] = 0$$

→
Because Markov and state- and date-contingent decisions

$$\lambda_1 = \beta E_1 [\lambda_2 (1+r_2)]$$

$$\mathbf{c}_2: \quad \beta^2 E_0 u'(c_2) - \beta^2 E_0 \lambda_2 = 0$$

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—————→
Holds for each
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—————→
Holds for each
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$$u'(c_2^j) = \lambda_2^j, \quad \forall j \in S$$

- One-period-ahead conditional expectation governs stochastic Euler condition

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STOCHASTIC – MARKOV SOLUTION

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- Solution of infinite-horizon consumer problem is a consumption **decision rule** $c(a, z; \cdot)$, asset **decision rule** $a(a, z; \cdot)$, and **value function** $V(a, z; \cdot)$ that satisfies

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- **(Stochastic) Euler equation**

$$u'(c(a, z)) = \beta E[u'(c(a', z'))(1 + r')]$$

- which is the **(expectational) TVC** in the limit $t \rightarrow \infty$:

$$\lim_{t \rightarrow \infty} E_0 \beta^t u'(c(a, z)) \cdot a(a, z) = 0$$

- **Budget constraint**

$$y + (1 + r)a - c(a, z) - a(a, z) = 0$$

- **Bellman Equation**

$$V(a, z; \cdot) \equiv u(c(a, z)) + \lambda (y + (1 + r)a - c(a, z) - a(a, z)) + \beta \cdot EV(a(a, z), z(a, z); \cdot)$$

taking as given (y, a, r) and **(Markov) transition function for $z \rightarrow z'$**

BELLMAN EQUATION

□ Bellman Equation (for $T \rightarrow \infty$)

$$V(a, z; \cdot) \equiv \max_{c, a'} \left\{ u(c) + \lambda \left(y + (1+r)a - c - a' \right) + \beta \cdot EV(a', z'; \cdot) \right\}$$

□ Use to characterize optimal decisions

Expectation in
Bellman Equation

Transition from
 $z \rightarrow z'$

□ Current-period FOCs, evaluated using $c(a, z; \cdot)$, $a(a, z; \cdot)$

c :

a' :

Env:

BELLMAN EQUATION

□ **Bellman Equation (for $T \rightarrow \infty$)**

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□ **Use to characterize optimal decisions**

Expectation in
Bellman Equation

Transition from
 $z \rightarrow z'$

□ **Current-period FOCs, evaluated using $c(a, z; \cdot), a(a, z; \cdot)$**

c: $u'(c(a, z)) - \lambda = 0$

a': $-\lambda + \beta EV_1(a(a, z), z(a, z); \cdot) = 0$

Env: $EV_1(a, z; \cdot) = \lambda(1+r)$

$u'(c(a, z)) = \beta E[u'(c(a, z))(1+r)]$

□ **Bellman analysis goes through as in deterministic case**

□ **(Given further technical conditions – see SLP)**

MARKOV RISK

- ❑ **Why does Markov assumption make everything work?**
- ❑ **Main issue in moving from deterministic dynamic programming to stochastic dynamic programming: **preserving recursivity****
 - ❑ **So exogenous states must also have recursive structure**
- ❑ **Shocks that have this recursive structure are Markov processes**
- ❑ **Markov has property that given the current realization, future realizations are independent of the past**
 - ❑ **“Limited history dependence”**
 - ❑ **“Finite memory”**

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- ❑ **In environments in which the “regularity conditions” that ensure standard Bellman analysis applies to stochastic problems are **not satisfied...****
- ❑ **...often simply need to ASSUME decision rules are Markov to make progress**