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# **LINEAR APPROXIMATION OF THE BASELINE RBC MODEL**

**MARCH 7, 2017**

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# LINEARIZATION

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- For  $f(x, y, z) = 0$ , multivariable Taylor linear expansion around  $(\bar{x}, \bar{y}, \bar{z})$

$$f(x, y, z) \approx f(\bar{x}, \bar{y}, \bar{z}) + f_x(\bar{x}, \bar{y}, \bar{z})(x - \bar{x}) + f_y(\bar{x}, \bar{y}, \bar{z})(y - \bar{y}) + f_z(\bar{x}, \bar{y}, \bar{z})(z - \bar{z})$$

(Illustrative example  
in scalars)

# LINEARIZATION OF THE RBC MODEL

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- Four equations describe the dynamic solution to RBC model

(Illustrative example  
in scalars)

- Consumption-leisure efficiency condition

$$-\frac{u_n(c_t, n_t)}{u_c(c_t, n_t)} = z_t m_n(k_t, n_t)$$

- Consumption-investment efficiency condition

$$u_c(c_t, n_t) = \beta E_t \left[ u_c(c_{t+1}, n_{t+1}) (1 - \delta + z_{t+1} m_k(k_{t+1}, n_{t+1})) \right]$$

- Aggregate resource constraint

$$c_t + k_{t+1} - (1 - \delta)k_t = z_t m(k_t, n_t)$$

- Law of motion for TFP

$$\ln z_{t+1} = (1 - \rho_z) \ln \bar{z} + \rho_z \ln z_t + \varepsilon_{t+1}^z$$

# STEADY STATE

- **Deterministic** steady state the natural local point of approximation
- Shut down all shocks and set exogenous variables at their means
- **The Idea:** Let economy run for many (infinite) periods
  - Time eventually “doesn’t matter” any more
  - Drop all time indices

$$-\frac{u_n(\bar{c}, \bar{n})}{u_c(\bar{c}, \bar{n})} = \bar{z}m_n(\bar{k}, \bar{n})$$

$$u_c(\bar{c}, \bar{n}) = \beta u_c(\bar{c}, \bar{n}) [m_k(\bar{k}, \bar{n}) + 1 - \delta]$$

$$\bar{c} + \delta\bar{k} = \bar{z}m(\bar{k}, \bar{n})$$

$$\ln \bar{z} = (1 - \rho_z) \ln \bar{z} + \rho_z \ln \bar{z} \Rightarrow \bar{z} = \bar{z} \quad (\text{a parameter of the model})$$

- Given functional forms and parameter values, solve for  $(c, n, k)$ 
  - **The steady state of the model**
  - **Taylor expansion around this point**

# LINEARIZATION ALGORITHMS

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- Schmitt-Grohe and Uribe (2004 *JEDC*)
  - A **perturbation** algorithm
    - A class of methods used to find an **approximate** solution to a problem that cannot be solved exactly, **by starting from the exact solution of a related problem**
    - Applicable if the problem can be formulated by adding a “small” term to the description of the exactly-solvable problem
  - (Matlab code available through Columbia Dept. of Economics web site – **DO NOT USE IN THIS CLASS!**)

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  - (Matlab code available through Columbia Dept. of Economics web site – **DO NOT USE IN THIS CLASS!**)
  
- **Uhlig (1999, chapter in *Computational Methods for the Study of Dynamic Economies*)**
  - Uses a generalized eigen-decomposition
    - Typically implemented with Schur decomposition (Sims algorithm)
  - Matlab code available at  
<http://www2.wiwi.hu-berlin.de/institute/wpol/html/toolkit.htm>

# LINEARIZATION OF THE RBC MODEL

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Define **co-state** vector and **state** vector

$$y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix} \quad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$

# LINEARIZATION OF THE RBC MODEL

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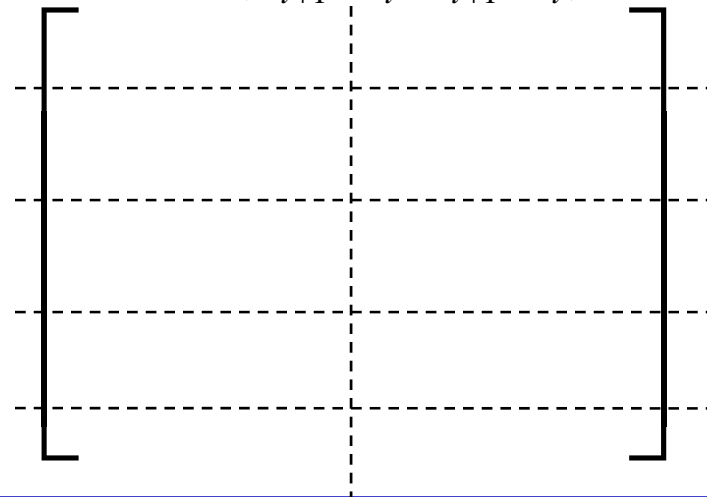
Order model's dynamic equations in a **vector**  $\equiv f(y_{t+1}, y_t, x_{t+1}, x_t) = 0$

Consumption-leisure efficiency condition

Consumption-investment efficiency condition

Aggregate resource constraint

Law of motion for TFP





# LINEARIZATION OF THE RBC MODEL

Need four **matrices** of derivatives

1. Differentiate  $f(y_{t+1}, y_t, x_{t+1}, x_t)$  with respect to (elements of)  $y_{t+1}$

First derivatives with respect to:

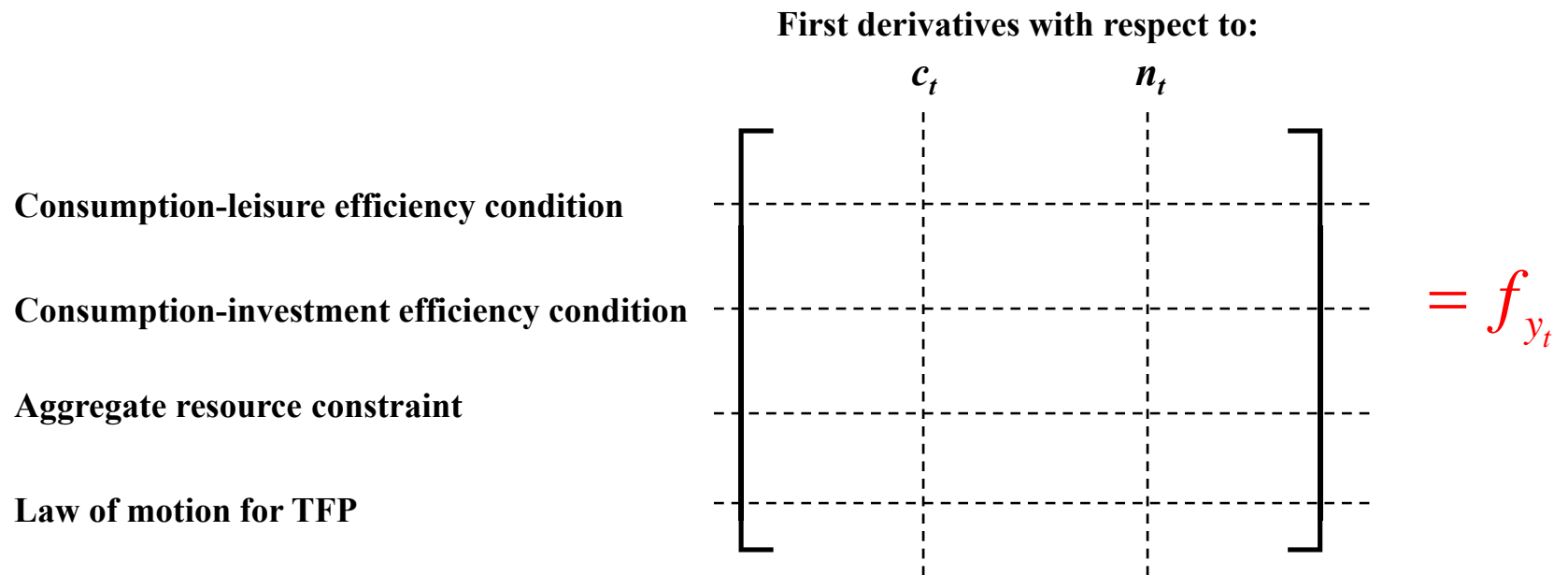
	$c_{t+1}$	$n_{t+1}$	
Consumption-leisure efficiency condition			
Consumption-investment efficiency condition			
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Law of motion for TFP			

$= f_{y_{t+1}}$

# LINEARIZATION OF THE RBC MODEL

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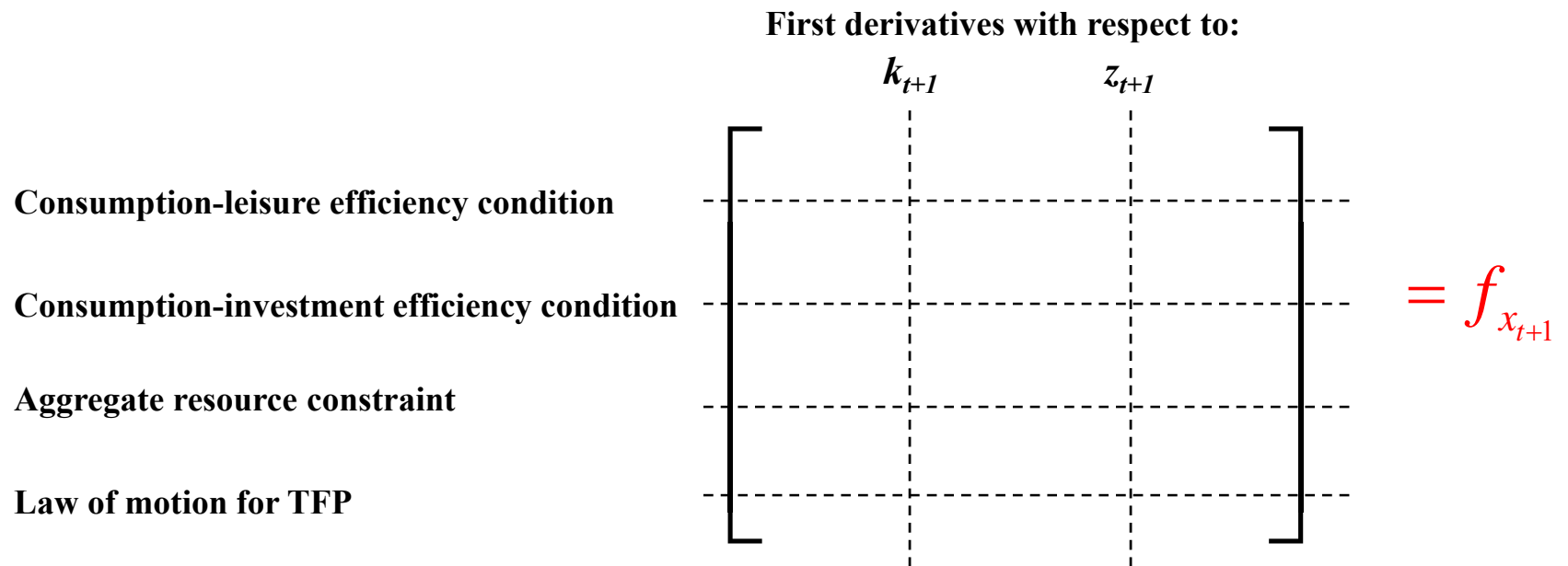
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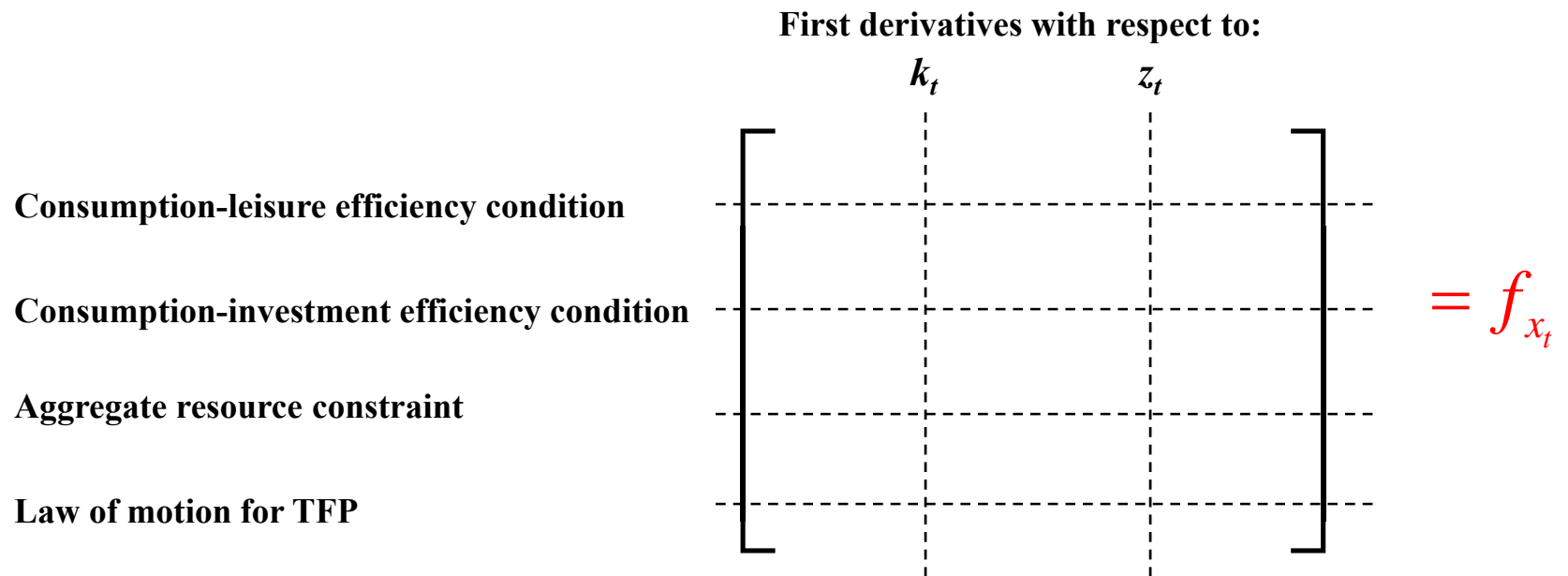
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# LINEARIZATION OF THE RBC MODEL

Need four **matrices** of derivatives

4. Differentiate  $f(y_{t+1}, y_t, x_{t+1}, x_t)$  with respect to (elements of)  $x_t$



# LINEARIZATION OF THE RBC MODEL

The model's dynamic **expectational** equations

$$E_t [f(y_{t+1}, y_t, x_{t+1}, x_t)] = E_t \begin{bmatrix} f^1(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^2(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^3(y_{t+1}, y_t, x_{t+1}, x_t) \\ f^4(y_{t+1}, y_t, x_{t+1}, x_t) \end{bmatrix}$$

**Consumption-leisure efficiency condition**  
**Consumption-investment efficiency condition**  
**Aggregate resource constraint**  
**Law of motion for TFP**

# LINEARIZATION OF THE RBC MODEL

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**Consumption-leisure efficiency condition**  
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## Conjecture equilibrium decision rules

**Note: g(.) and h(.) are time invariant functions!**

$$y_t = g(x_t, \sigma)$$

$$x_{t+1} = h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}$$

Substitute decision rules  
into dynamic equations

“Perturbation parameter”:  
governs size of shocks

Matrix of standard  
deviations of state  
variables

# LINEARIZATION OF THE RBC MODEL

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The model's dynamic **expectational** equations

$$\begin{aligned} E_t [f(y_{t+1}, y_t, x_{t+1}, x_t)] &= 0 \\ &= E_t [f(g(x_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, x_t)] \\ &= E_t [f(g(h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, x_t)] \\ &\equiv F(x_t, \sigma) \end{aligned}$$

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 \end{aligned}$$



$$F_x(x_t, \sigma) = 0 \quad F_\sigma(x_t, \sigma) = 0$$



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Using chain rule and  
suppressing arguments

$$F_x(x_t, \sigma) =$$

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$$F_x(x_t, \sigma) = f_{y_{t+1}} \cdot g_x \cdot h_x$$

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$$F_x(x_t, \sigma) = f_{y_{t+1}} \cdot g_x \cdot h_x + f_{y_t} \cdot g_x$$

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 &= 0
 \end{aligned}$$

Setting  $\sigma = 0$  shuts  
down shocks

Holds, in particular, at the **deterministic** steady state  $(\bar{x}, 0)$

$$F_x(\bar{x}, 0) = f_{y_{t+1}} \cdot g_x \cdot h_x + f_{y_t} \cdot g_x + f_{x_{t+1}} \cdot h_x + f_{x_t} = 0$$

Each term is evaluated at  
the steady state – just as  
Taylor theorem requires

# LINEARIZATION OF THE RBC MODEL

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- A **quadratic** equation in the elements of  $g_x$  and  $h_x$  evaluated at the steady state

$$F_x(\bar{x}, 0) = f_{y_{t+1}}(\bar{x}, 0) \cdot g_x(\bar{x}, 0) \cdot h_x(\bar{x}, 0) + f_{y_t}(\bar{x}, 0) \cdot g_x(\bar{x}, 0) + f_{x_{t+1}}(\bar{x}, 0) \cdot h_x(\bar{x}, 0) + f_{x_t}(\bar{x}, 0) = 0$$

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$$y_t = g(x_t, \sigma) \approx g(\bar{x}, 0) + g_x(\bar{x}, 0)(x_t - \bar{x}) + \overset{= 0}{g_\sigma(\bar{x}, 0)\sigma} \quad \text{SGU Theorem 1:}$$

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- **DONE!!!**

- Now conduct impulse responses, tabulate business cycle moments, write paper

# CERTAINTY EQUIVALENCE

---

- Displayed by a model if decision rules do **not** depend on the standard deviation of exogenous uncertainty – e.g., **PRECAUTIONARY SAVINGS!**
- For **stochastic** problems with **quadratic objective function** and **linear constraints**, the decision rules are identical to those of the nonstochastic problem

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- Here, we have

$$y_t = g(x_t, \sigma) \approx g(\bar{x}, 0) + g_x(\bar{x}, 0)(x_t - \bar{x}) + \cancel{g_\sigma(\bar{x}, 0)\sigma} = 0$$

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- **SGU Theorem 1:  $g_\sigma = 0$  and  $h_\sigma = 0$** 
  - First-order approximated decision rules do not depend on the size of the shocks, which is governed by  $\sigma$
  - Not the same thing as “**exact CE,**” but refer to it as CE

## LINEARIZING THE RBC MODEL

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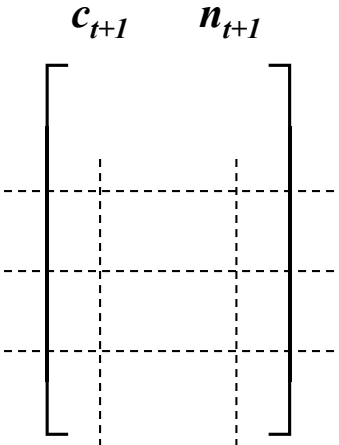
- Assume  $u(c_t, n_t) = \ln c_t - \psi \ln n_t$  and  $m(k_t, n_t) = k_t^\alpha n_t^{1-\alpha}$
- $\therefore$  consumption-leisure efficiency condition is  $\frac{\psi c_t}{n_t} - (1 - \alpha) z_t k_t^\alpha n_t^{-\alpha} = 0$

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- Let  $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \frac{\psi c_t}{n_t} - (1 - \alpha) z_t k_t^\alpha n_t^{-\alpha} = 0$  (and recall  $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$   $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ )

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  - Compute first row of matrix  $f_{yt+1}$ 
    - Consumption-leisure efficiency condition
    - Consumption-investment efficiency condition
    - Aggregate resource constraint
    - Law of motion for TFP
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- |   | $c_{t+1}$ | $n_{t+1}$ |
|---|-----------|-----------|
| Consumption-leisure efficiency condition    | 0         |           |
| Consumption-investment efficiency condition |           |           |
| Aggregate resource constraint               |           |           |
| Law of motion for TFP                       |           |           |



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	$c_{t+1}$	$n_{t+1}$
Consumption-leisure efficiency condition	0	0
Consumption-investment efficiency condition	-----	-----
Aggregate resource constraint	-----	-----
Law of motion for TFP	-----	-----

# LINEARIZING THE RBC MODEL

- Assume  $u(c_t, n_t) = \ln c_t - \psi \ln n_t$  and  $m(k_t, n_t) = k_t^\alpha n_t^{1-\alpha}$
- $\therefore$  consumption-leisure efficiency condition is  $\frac{\psi c_t}{n_t} - (1 - \alpha) z_t k_t^\alpha n_t^{-\alpha} = 0$

- Let  $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \frac{\psi c_t}{n_t} - (1 - \alpha) z_t k_t^\alpha n_t^{-\alpha} = 0$  (and recall  $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$   $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ )

- Compute first row of matrix  $f_{yt}$

	$c_t$	$n_t$
Consumption-leisure efficiency condition	$\left[ \begin{array}{cc} \frac{\psi}{n_t} & - \frac{\psi c_t}{n_t^2} + \alpha (1 - \alpha) z_t k_t^\alpha n_t^{-\alpha - 1} \end{array} \right]$	
Consumption-investment efficiency condition	-----	
Aggregate resource constraint	-----	
Law of motion for TFP	-----	

# LINEARIZING THE RBC MODEL

- Assume  $u(c_t, n_t) = \ln c_t - \psi \ln n_t$  and  $m(k_t, n_t) = k_t^\alpha n_t^{1-\alpha}$
- $\therefore$  consumption-leisure efficiency condition is  $\frac{\psi c_t}{n_t} - (1 - \alpha) z_t k_t^\alpha n_t^{-\alpha} = 0$
- Let  $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \frac{\psi c_t}{n_t} - (1 - \alpha) z_t k_t^\alpha n_t^{-\alpha} = 0$  (and recall  $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$   $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ )
- Compute first row of matrix  $f_{xt+1}$ 
  - Consumption-leisure efficiency condition
  - Consumption-investment efficiency condition
  - Aggregate resource constraint
  - Law of motion for TFP

$$\begin{array}{cc}
 & k_{t+1} & z_{t+1} \\
 \left[ \begin{array}{cc}
 \mathbf{O} & \mathbf{O} \\
 \text{---} & \text{---} \\
 \text{---} & \text{---} \\
 \text{---} & \text{---}
 \end{array} \right]
 \end{array}$$

# LINEARIZING THE RBC MODEL

- Assume  $u(c_t, n_t) = \ln c_t - \psi \ln n_t$  and  $m(k_t, n_t) = k_t^\alpha n_t^{1-\alpha}$
- $\therefore$  consumption-leisure efficiency condition is  $\frac{\psi c_t}{n_t} - (1-\alpha)z_t k_t^\alpha n_t^{-\alpha} = 0$

- Let  $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \frac{\psi c_t}{n_t} - (1-\alpha)z_t k_t^\alpha n_t^{-\alpha} = 0$  (and recall  $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$   $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ )

- Compute first row of matrix  $f_{xt}$

Consumption-leisure efficiency condition

Consumption-investment efficiency condition

Aggregate resource constraint

Law of motion for TFP

$$\begin{bmatrix} k_t & & z_t & \\ -\alpha(1-\alpha)z_t \frac{k_t^{\alpha-1}}{n_t^\alpha} & & -(1-\alpha) \frac{k_t^\alpha}{n_t^{\alpha+1}} & \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

# LINEARIZING THE RBC MODEL

- In deterministic steady state, the first rows of  $f_{y_{t+1}}, f_{y_t}, f_{x_{t+1}}, f_{x_t}$  are

$f_{y_{t+1}}$	0	0
$f_{y_t}$	$\frac{\psi}{\bar{n}}$	$-\frac{\psi \bar{c}}{\bar{n}^2} + \alpha(1-\alpha)\bar{z}\bar{k}^\alpha \bar{n}^{-\alpha-1}$
$f_{x_{t+1}}$	0	0
$f_{x_t}$	$-\alpha(1-\alpha)\bar{z}\frac{\bar{k}^{\alpha-1}}{\bar{n}^\alpha}$	$-(1-\alpha)\frac{\bar{k}^\alpha}{\bar{n}^{\alpha+1}}$

# LINEARIZING THE RBC MODEL

- In deterministic steady state, the first rows of  $f_{yt+1}, f_{yt}, f_{xt+1}, f_{xt}$  are

$$\begin{array}{l}
 f_{yt+1} \qquad \qquad \qquad 0 \qquad \qquad \qquad 0 \\
 \\
 f_{yt} \qquad \qquad \qquad \frac{\psi}{\bar{n}} \qquad \qquad \qquad -\frac{\psi \bar{c}}{\bar{n}^2} + \alpha(1-\alpha)\bar{z}\bar{k}^\alpha \bar{n}^{-\alpha-1} \\
 \\
 f_{xt+1} \qquad \qquad \qquad 0 \qquad \qquad \qquad 0 \\
 \\
 f_{xt} \qquad \qquad \qquad -\alpha(1-\alpha)\bar{z} \frac{\bar{k}^{\alpha-1}}{\bar{n}^\alpha} \qquad \qquad \qquad -(1-\alpha) \frac{\bar{k}^\alpha}{\bar{n}^{\alpha+1}}
 \end{array}$$

- How to compute derivatives  $f_{yt+1}, f_{yt}, f_{xt+1}, f_{xt}$ ?
  - By hand (feasible for small models)
  - Schmitt-Grohe and Uribe Matlab analytical routines
  - Your own Maple or Mathematica programs
  - **MuPad**
  - Dynare package