Linear Approximation of the Baseline RBC Model

March 7, 2017

LINEARIZATION

- For $f(x, y, z) = 0$, multivariable Taylor linear expansion around $(\overline{x}, \overline{y}, \overline{z})$

$$f(x, y, z) \approx f(\overline{x}, \overline{y}, \overline{z}) + f_x(\overline{x}, \overline{y}, \overline{z})(x - \overline{x}) + f_y(\overline{x}, \overline{y}, \overline{z})(y - \overline{y}) + f_z(\overline{x}, \overline{y}, \overline{z})(z - \overline{z})$$

(Illustrative example in scalars)

LINEARIZATION OF THE RBC MODEL

- For $f(x,y,z)=0$, multivariable Taylor linear expansion around $(\bar{x},\bar{y},\bar{z})$

$$f(x,y,z) \approx f(\bar{x},\bar{y},\bar{z}) + f_x(\bar{x},\bar{y},\bar{z})(x-\bar{x}) + f_y(\bar{x},\bar{y},\bar{z})(y-\bar{y}) + f_z(\bar{x},\bar{y},\bar{z})(z-\bar{z})$$

- Four equations describe the dynamic solution to RBC model

(Illustrative example in scalars)

  - Consumption-leisure efficiency condition

$$-\frac{u_n(c_t,n_t)}{u_c(c_t,n_t)} = z_t m_n(k_t,n_t)$$

  - Consumption-investment efficiency condition

$$u_c(c_t,n_t) = \beta E_t\left[u_c(c_{t+1},n_{t+1})\left(1-\delta+z_{t+1}m_k(k_{t+1},n_{t+1})\right)\right]$$

  - Aggregate resource constraint

$$c_t + k_{t+1} - (1-\delta)k_t = z_t m(k_t,n_t)$$

  - Law of motion for TFP

$$\ln z_{t+1} = (1-\rho_z)\ln\bar{z} + \rho_z \ln z_t + \varepsilon^z_{t+1}$$

STEADY STATE

- **Deterministic steady state the natural local point of approximation**
- **Shut down all shocks and set exogenous variables at their means**
- **The Idea: Let economy run for many (infinite) periods**
  - **Time eventually "doesn't matter" any more**
  - **Drop all time indices**

$$-\frac{u_n(\bar{c},\bar{n})}{u_c(\bar{c},\bar{n})} = \bar{z}m_n(\bar{k},\bar{n})$$

$$u_c(\bar{c},\bar{n}) = \beta u_c(\bar{c},\bar{n})\left[m_k(\bar{k},\bar{n}) + 1 - \delta\right]$$

$$\bar{c} + \delta\bar{k} = \bar{z}m(\bar{k},\bar{n})$$

$$\ln\bar{z} = (1-\rho_z)\ln\bar{z} + \rho_z\ln\bar{z} \Rightarrow \bar{z} = \bar{z} \quad \text{(a parameter of the model)}$$

- **Given functional forms and parameter values, solve for (*c*, *n*, *k*)**
  - **The steady state of the model**
  - **Taylor expansion around this point**

LINEARIZATION ALGORITHMS

- Schmitt-Grohe and Uribe (2004 *JEDC*)
  - A perturbation algorithm
    - A class of methods used to find an approximate solution to a problem that cannot be solved exactly, by starting from the exact solution of a related problem
    - Applicable if the problem can be formulated by adding a “small” term to the description of the exactly-solvable problem
  - (Matlab code available through Columbia Dept. of Economics web site – DO NOT USE IN THIS CLASS!)

LINEARIZATION ALGORITHMS

- **Schmitt-Grohe and Uribe (2004 *JEDC*)**
  - **A perturbation algorithm**
    - **A class of methods used to find an approximate solution to a problem that cannot be solved exactly, by starting from the exact solution of a related problem**
    - **Applicable if the problem can be formulated by adding a "small" term to the description of the exactly-solvable problem**
  - **(Matlab code available through Columbia Dept. of Economics web site – DO NOT USE IN THIS CLASS!)**

- **Uhlig (1999, chapter in *Computational Methods for the Study of Dynamic Economies*)**
  - **Uses a generalized eigen-decomposition**
    - **Typically implemented with Schur decomposition (Sims algorithm)**
  - **Matlab code available at**

    http://www2.wiwi.hu-berlin.de/institute/wpol/html/toolkit.htm

LINEARIZATION OF THE RBC MODEL

Define co-state vector and state vector

$$y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix} \qquad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$

LINEARIZATION OF THE RBC MODEL

Define co-state vector and state vector

$$y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix} \qquad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$

Order model's dynamic equations in a vector $\equiv f(y_{t+1}, y_t, x_{t+1}, x_t) = 0$

Consumption-leisure efficiency condition

Consumption-investment efficiency condition

Aggregate resource constraint

Law of motion for TFP

LINEARIZATION OF THE RBC MODEL

<u>Need four matrices of derivatives</u>

1. Differentiate $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** y_{t+1}

First derivatives with respect to:

$$\begin{array}{l c} & \begin{array}{cc} c_{t+1} & n_{t+1} \end{array} \\ \begin{array}{l} \text{Consumption-leisure efficiency condition} \\ \text{Consumption-investment efficiency condition} \\ \text{Aggregate resource constraint} \\ \text{Law of motion for TFP} \end{array} & \left[\begin{array}{cc} \; & \; \\ \; & \; \\ \; & \; \\ \; & \; \end{array} \right] = f_{y_{t+1}} \end{array}$$

LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

2. Differentiate $f\left(y_{t+1}, y_t, x_{t+1}, x_t\right)$ **with respect to (elements of)** y_t

First derivatives with respect to:

$$\begin{array}{l c} & \begin{array}{cc} \boldsymbol{c_t} \quad & \quad \boldsymbol{n_t} \end{array} \\ \begin{array}{l} \textbf{Consumption-leisure efficiency condition} \\ \textbf{Consumption-investment efficiency condition} \\ \textbf{Aggregate resource constraint} \\ \textbf{Law of motion for TFP} \end{array} & \left[\begin{array}{cc} \quad & \quad \\ \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{array}\right] \end{array} = f_{y_t}$$

LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

3. Differentiate $f\left(y_{t+1}, y_t, x_{t+1}, x_t\right)$ **with respect to (elements of)** x_{t+1}

First derivatives with respect to:

$$\begin{array}{l c}
 & \begin{matrix} k_{t+1} & \qquad z_{t+1} \end{matrix} \\
\begin{matrix} \textbf{Consumption-leisure efficiency condition} \\ \textbf{Consumption-investment efficiency condition} \\ \textbf{Aggregate resource constraint} \\ \textbf{Law of motion for TFP} \end{matrix} & \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix} = f_{x_{t+1}}
\end{array}$$

LINEARIZATION OF THE RBC MODEL

Need four matrices of derivatives

4. Differentiate $f(y_{t+1}, y_t, x_{t+1}, x_t)$ **with respect to (elements of)** x_t

First derivatives with respect to:

$$
\begin{array}{l c}
 & \begin{array}{cc} k_t & z_t \end{array} \\
\begin{array}{l}
\textbf{Consumption-leisure efficiency condition} \\
\textbf{Consumption-investment efficiency condition} \\
\textbf{Aggregate resource constraint} \\
\textbf{Law of motion for TFP}
\end{array} &
\begin{bmatrix} & \\ & \\ & \\ & \end{bmatrix}
\end{array}
= f_{x_t}
$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]=E_t\begin{bmatrix} f^1(y_{t+1},y_t,x_{t+1},x_t) \\ f^2(y_{t+1},y_t,x_{t+1},x_t) \\ f^3(y_{t+1},y_t,x_{t+1},x_t) \\ f^4(y_{t+1},y_t,x_{t+1},x_t) \end{bmatrix}$$

Consumption-leisure efficiency condition

Consumption-investment efficiency condition

Aggregate resource constraint

Law of motion for TFP

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]=E_t\begin{bmatrix} f^1(y_{t+1},y_t,x_{t+1},x_t) \\ f^2(y_{t+1},y_t,x_{t+1},x_t) \\ f^3(y_{t+1},y_t,x_{t+1},x_t) \\ f^4(y_{t+1},y_t,x_{t+1},x_t) \end{bmatrix}$$

Consumption-leisure efficiency condition
Consumption-investment efficiency condition
Aggregate resource constraint
Law of motion for TFP

Conjecture equilibrium decision rules

Note: g(.) and h(.) are time invariant functions!

$$y_t = g(x_t,\sigma)$$
$$x_{t+1} = h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1}$$

"Perturbation parameter": governs size of shocks

Matrix of standard deviations of state variables

Substitute decision rules into dynamic equations

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t)\right] = 0$$

$$= E_t\left[f(g(x_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, x_t\right]$$

$$= E_t\left[f(g(h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, x_t\right]$$

$$\equiv F(x_t, \sigma)$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t)\right] = 0$$

$$= E_t\left[f(g(x_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, x_t\right]$$

$$= E_t\left[f(g(h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, x_t\right]$$

$$\equiv F(x_t, \sigma)$$

$$\Downarrow$$

$$F_x(x_t, \sigma) = 0 \qquad F_\sigma(x_t, \sigma) = 0$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t)\right] = 0$$

$$= E_t\left[f(g(x_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}, x_t\right]$$

$$= E_t\left[f(g(h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}, x_t\right]$$

$$\equiv F(x_t,\sigma)$$

$$\Downarrow$$

Using chain rule and suppressing arguments

$$F_x(x_t,\sigma) =$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$
\begin{aligned}
E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t)\right] = 0 \\
&= E_t\left[f(g(x_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}, x_t\right] \\
&= E_t\left[f(g(h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}, x_t\right] \\
&\equiv F(x_t,\sigma)
\end{aligned}
$$

$$\Downarrow$$

Using chain rule and suppressing arguments

$$F_x(x_t,\sigma) = f_{y_{t+1}} \cdot g_x \cdot h_x$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t)\right] = 0$$

$$= E_t\left[f(g(x_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1}, x_t\right]$$

$$= E_t\left[f(g(h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1}, x_t\right]$$

$$\equiv F(x_t,\sigma)$$

$$\Downarrow$$

Using chain rule and suppressing arguments

$$F_x(x_t,\sigma) = f_{y_{t+1}} \cdot g_x \cdot h_x + f_{y_t} \cdot g_x$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t) \right] = 0$$

$$= E_t\left[f(g(x_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}, x_t \right]$$

$$= E_t\left[f(g(h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}, x_t \right]$$

$$\equiv F(x_t,\sigma)$$

$$\Downarrow$$

Using chain rule and suppressing arguments

$$F_x(x_t,\sigma) = f_{y_{t+1}} \cdot g_x \cdot h_x + f_{y_t} \cdot g_x + f_{x_{t+1}} \cdot h_x$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1},y_t,x_{t+1},x_t)\right]=0$$
$$=E_t\left[f(g(x_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$
$$=E_t\left[f(g(h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},\sigma),g(x_t,\sigma),h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},x_t\right]$$
$$\equiv F(x_t,\sigma)$$

$$\Downarrow$$

Using chain rule and suppressing arguments

$$F_x(x_t,\sigma)=f_{y_{t+1}}\cdot g_x\cdot h_x+f_{y_t}\cdot g_x+f_{x_{t+1}}\cdot h_x+f_{x_t}$$
$$=0$$

LINEARIZATION OF THE RBC MODEL

The model's dynamic expectational equations

$$E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t)\right] = 0$$

$$= E_t\left[f(g(x_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1}, x_t\right]$$

$$= E_t\left[f(g(h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1}, x_t\right]$$

$$\equiv F(x_t,\sigma)$$

$$\Downarrow$$

Using chain rule and suppressing arguments

$$F_x(x_t,\sigma) = f_{y_{t+1}} \cdot g_x \cdot h_x + f_{y_t} \cdot g_x + f_{x_{t+1}} \cdot h_x + f_{x_t}$$

$$= 0$$

Setting $\sigma = 0$ shuts down shocks

Holds, in particular, at the deterministic steady state $(\bar{x}, 0)$

$$F_x(\bar{x},0) = f_{y_{t+1}} \cdot g_x \cdot h_x + f_{y_t} \cdot g_x + f_{x_{t+1}} \cdot h_x + f_x = 0$$

Each term is evaluated at the steady state – just as Taylor theorem requires

LINEARIZATION OF THE RBC MODEL

- **A quadratic equation in the elements of g_x and h_x evaluated at the steady state**

$$F_x(\overline{x},0) = f_{y_{t+1}}(\overline{x},0)\cdot g_x(\overline{x},0)\cdot h_x(\overline{x},0) + f_{y_t}(\overline{x},0)\cdot g_x(\overline{x},0) + f_{x_{t+1}}(\overline{x},0)\cdot h_x(\overline{x},0) + f_{x_t}(\overline{x},0) = 0$$

- **Solve numerically for the elements of g_x and h_x (use `fsolve` OR conduct an eigenvalue decomposition in Matlab)**

LINEARIZATION OF THE RBC MODEL

- **A quadratic equation in the elements of g_x and h_x evaluated at the steady state**

$$F_x(\bar{x},0) = f_{y_{t+1}}(\bar{x},0)\cdot g_x(\bar{x},0)\cdot h_x(\bar{x},0) + f_{y_t}(\bar{x},0)\cdot g_x(\bar{x},0) + f_{x_{t+1}}(\bar{x},0)\cdot h_x(\bar{x},0) + f_{x_t}(\bar{x},0) = 0$$

- **Solve numerically for the elements of g_x and h_x (use `fsolve` OR conduct an eigenvalue decomposition in Matlab)**
- **Recall conjectured equilibrium decision rules**

$$y_t = g(x_t, \sigma)$$

$$x_{t+1} = h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}$$

LINEARIZATION OF THE RBC MODEL

- A quadratic equation in the elements of g_x and h_x evaluated at the steady state

$$F_x(\bar{x},0) = f_{y_{t+1}}(\bar{x},0)\cdot g_x(\bar{x},0)\cdot h_x(\bar{x},0) + f_{y_t}(\bar{x},0)\cdot g_x(\bar{x},0) + f_{x_{t+1}}(\bar{x},0)\cdot h_x(\bar{x},0) + f_{x_t}(\bar{x},0) = 0$$

- Solve numerically for the elements of g_x and h_x (use `fsolve` OR conduct an eigenvalue decomposition in Matlab)
- Recall conjectured equilibrium decision rules

$$y_t = g(x_t,\sigma)$$

$$x_{t+1} = h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}$$

- First-order approximation is

$$y_t = g(x_t,\sigma) \approx g(\bar{x},0) + g_x(\bar{x},0)(x_t-\bar{x}) + \overbrace{g_\sigma}^{=0}(\bar{x},0)\sigma$$

$$x_{t+1} = h(x_t,\sigma) \approx h(\bar{x},0) + h_x(\bar{x},0)(x_t-\bar{x}) + \underbrace{h_\sigma}_{=0}(\bar{x},0)\sigma$$

SGU Theorem 1:

$g_\sigma = 0$ and $h_\sigma = 0$

LINEARIZATION OF THE RBC MODEL

- A quadratic equation in the elements of g_x and h_x evaluated at the steady state

$$F_x(\bar{x},0) = f_{y_{t+1}}(\bar{x},0)\cdot g_x(\bar{x},0)\cdot h_x(\bar{x},0) + f_{y_t}(\bar{x},0)\cdot g_x(\bar{x},0) + f_{x_{t+1}}(\bar{x},0)\cdot h_x(\bar{x},0) + f_{x_t}(\bar{x},0) = 0$$

- Solve numerically for the elements of g_x and h_x (use `fsolve` OR conduct an eigenvalue decomposition in Matlab)
- Recall conjectured equilibrium decision rules

$$y_t = g(x_t,\sigma)$$
$$x_{t+1} = h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}$$

- First-order approximation is

$$y_t = g(x_t,\sigma) \approx g(\bar{x},0) + g_x(\bar{x},0)(x_t - \bar{x}) + \overbrace{g_\sigma}^{=0}(\bar{x},0)\sigma$$
$$x_{t+1} = h(x_t,\sigma) \approx h(\bar{x},0) + h_x(\bar{x},0)(x_t - \bar{x}) + \underbrace{h_\sigma}_{=0}(\bar{x},0)\sigma$$

SGU Theorem 1:

$g_\sigma = 0$ and $h_\sigma = 0$

- DONE!!!
  - Now conduct impulse responses, tabulate business cycle moments, write paper

CERTAINTY EQUIVALENCE

- **Displayed by a model if decision rules do not depend on the standard deviation of exogenous uncertainty – e.g., PRECAUTIONARY SAVINGS!**
- **For stochastic problems with quadratic objective function and linear constraints, the decision rules are identical to those of the nonstochastic problem**

CERTAINTY EQUIVALENCE

- **Displayed by a model if decision rules do not depend on the standard deviation of exogenous uncertainty – e.g., PRECAUTIONARY SAVINGS!**

- **For stochastic problems with quadratic objective function and linear constraints, the decision rules are identical to those of the nonstochastic problem**

- **Here, we have**

$$y_t = g(x_t, \sigma) \approx g(\bar{x}, 0) + g_x(\bar{x}, 0)(x_t - \bar{x}) + \overset{=0}{g_\sigma}(\bar{x}, 0)\sigma$$

$$x_{t+1} = h(x_t, \sigma) \approx h(\bar{x}, 0) + h_x(\bar{x}, 0)(x_t - \bar{x}) + \underset{=0}{h_\sigma}(\bar{x}, 0)\sigma$$

- **SGU Theorem 1: $g_\sigma = 0$ and $h_\sigma = 0$**
  - **First-order approximated decision rules do not depend on the size of the shocks, which is governed by σ**
  - **Not the same thing as "exact CE," but refer to it as CE**

LINEARIZING THE RBC MODEL

- **Assume** $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$

LINEARIZING THE RBC MODEL

- Assume $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$
- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $\quad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$**)**

LINEARIZING THE RBC MODEL

- **Assume** $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$
- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $\quad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ **)**
- **Compute first row of matrix** ***f_{yt+1}***

$$\begin{array}{l c} & \begin{array}{cc} \boldsymbol{c_{t+1}} & \boldsymbol{n_{t+1}} \end{array} \\ \begin{array}{l} \textbf{Consumption-leisure efficiency condition} \\ \textbf{Consumption-investment efficiency condition} \\ \textbf{Aggregate resource constraint} \\ \textbf{Law of motion for TFP} \end{array} & \left[\begin{array}{c:c} \quad & \quad \\ \hdashline & \\ \hdashline & \\ \hdashline & \end{array}\right] \end{array}$$

LINEARIZING THE RBC MODEL

- **Assume** $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$
- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ **)**
- **Compute first row of matrix f_{yt+1}**

$$\begin{array}{lcc} & \boldsymbol{c_{t+1}} & \boldsymbol{n_{t+1}} \\ \textbf{Consumption-leisure efficiency condition} & \left[\begin{array}{c} 0 \\ \\ \\ \\ \end{array}\right. & \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right] \\ \textbf{Consumption-investment efficiency condition} & & \\ \textbf{Aggregate resource constraint} & & \\ \textbf{Law of motion for TFP} & & \end{array}$$

LINEARIZING THE RBC MODEL

- Assume $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$
- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ **)**
- **Compute first row of matrix *f_{yt+1}***

$$
\begin{array}{l c}
 & \begin{array}{cc} c_{t+1} & n_{t+1} \end{array} \\
\begin{array}{l}
\textbf{Consumption-leisure efficiency condition} \\
\textbf{Consumption-investment efficiency condition} \\
\textbf{Aggregate resource constraint} \\
\textbf{Law of motion for TFP}
\end{array} &
\begin{bmatrix} 0 & 0 \\ & \\ & \\ & \end{bmatrix}
\end{array}
$$

LINEARIZING THE RBC MODEL

- Assume $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ and $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
- $\therefore$ consumption-leisure efficiency condition is $\dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$
- Let $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ (and recall $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$)
- Compute first row of matrix f_{yt}

$$\begin{array}{ll} & \begin{array}{cc} c_t & n_t \end{array} \\ \begin{array}{l} \text{Consumption-leisure efficiency condition} \\ \text{Consumption-investment efficiency condition} \\ \text{Aggregate resource constraint} \\ \text{Law of motion for TFP} \end{array} & \begin{bmatrix} \dfrac{\psi}{n_t} & -\dfrac{\psi c_t}{n_t^2} + \alpha(1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha-1} \\ & \\ & \\ & \end{bmatrix} \end{array}$$

LINEARIZING THE RBC MODEL

- **Assume** $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$
- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $\quad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ **)**
- **Compute first row of matrix** ***f_{xt+1}***

$$\begin{array}{lc}
 & \begin{array}{cc} k_{t+1} & z_{t+1} \end{array} \\
\begin{array}{l} \textbf{Consumption-leisure efficiency condition} \\ \textbf{Consumption-investment efficiency condition} \\ \textbf{Aggregate resource constraint} \\ \textbf{Law of motion for TFP} \end{array} & \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ & \\ & \\ & \end{bmatrix}
\end{array}$$

LINEARIZING THE RBC MODEL

- **Assume** $u(c_t, n_t) = \ln c_t - \psi \ln n_t$ **and** $m(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$
- $\therefore$ **consumption-leisure efficiency condition is** $\dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$
- **Let** $f^1(y_{t+1}, y_t, x_{t+1}, x_t) = \dfrac{\psi c_t}{n_t} - (1-\alpha) z_t k_t^{\alpha} n_t^{-\alpha} = 0$ **(and recall** $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$ $\quad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$ **)**
- **Compute first row of matrix** ***f_{xt}***

	k_t	z_t
Consumption-leisure efficiency condition	$-\alpha(1-\alpha) z_t \dfrac{k_t^{\alpha-1}}{n_t^{\alpha}}$	$-(1-\alpha)\dfrac{k_t^{\alpha}}{n_t^{\alpha+1}}$
Consumption-investment efficiency condition		
Aggregate resource constraint		
Law of motion for TFP		

LINEARIZING THE RBC MODEL

- **In deterministic steady state, the first rows of $f_{yt+1}, f_{yt}, f_{xt+1}, f_{xt}$ are**

f_{yt+1}	0	0
f_{yt}	$\dfrac{\psi}{\bar{n}}$	$-\dfrac{\psi \bar{c}}{\bar{n}^2} + \alpha(1-\alpha)\bar{z}\bar{k}^{\alpha}\bar{n}^{-\alpha-1}$
f_{xt+1}	0	0
f_{xt}	$-\alpha(1-\alpha)\bar{z}\dfrac{\bar{k}^{\alpha-1}}{\bar{n}^{\alpha}}$	$-(1-\alpha)\dfrac{\bar{k}^{\alpha}}{\bar{n}^{\alpha+1}}$

LINEARIZING THE RBC MODEL

- In deterministic steady state, the first rows of $f_{yt+1}, f_{yt}, f_{xt+1}, f_{xt}$ are

f_{yt+1}	0	0
f_{yt}	$\frac{\psi}{\bar{n}}$	$-\frac{\psi \bar{c}}{\bar{n}^2} + \alpha(1-\alpha)\bar{z}\bar{k}^{\alpha}\bar{n}^{-\alpha-1}$
f_{xt+1}	0	0
f_{xt}	$-\alpha(1-\alpha)\bar{z}\frac{\bar{k}^{\alpha-1}}{\bar{n}^{\alpha}}$	$-(1-\alpha)\frac{\bar{k}^{\alpha}}{\bar{n}^{\alpha+1}}$

- How to compute derivatives f_{yt+1}, f_{yt}, f_{xt+1}, f_{xt}?
  - By hand (feasible for small models)
  - Schmitt-Grohe and Uribe Matlab analytical routines
  - Your own Maple or Mathematica programs
  - MuPad
  - Dynare package