Second-Order Approximation

January 17, 2018

SECOND-ORDER PERTURBATIION

- **For $f(x,y,z)=0$, second-order Taylor expansion around $(\overline{x},\overline{y},\overline{z})$**

$$f(x,y,z) \approx f(\overline{x},\overline{y},\overline{z}) + f_x(\overline{x},\overline{y},\overline{z})(x-\overline{x}) + f_y(\overline{x},\overline{y},\overline{z})(y-\overline{y}) + f_z(\overline{x},\overline{y},\overline{z})(z-\overline{z})$$

(Illustrative example in scalars)

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$$
\begin{aligned}
f(x,y,z) \approx\ & f(\overline{x},\overline{y},\overline{z}) + f_x(\overline{x},\overline{y},\overline{z})(x-\overline{x}) + f_y(\overline{x},\overline{y},\overline{z})(y-\overline{y}) + f_z(\overline{x},\overline{y},\overline{z})(z-\overline{z}) \\
& + \frac{1}{2} f_{xx}(\overline{x},\overline{y},\overline{z})(x-\overline{x})^2 + \frac{1}{2} f_{yy}(\overline{x},\overline{y},\overline{z})(y-\overline{y})^2 + \frac{1}{2} f_{zz}(\overline{x},\overline{y},\overline{z})(z-\overline{z})^2 \\
& + \frac{1}{2} f_{xy}(\overline{x},\overline{y},\overline{z})(x-\overline{x})(y-\overline{y}) + \frac{1}{2} f_{yx}(\overline{x},\overline{y},\overline{z})(y-\overline{y})(x-\overline{x}) + \ldots.
\end{aligned}
$$

(Illustrative example in scalars)

- **Why a second-order approximation?**

SECOND-ORDER PERTURBATIION

- **For $f(x,y,z)=0$, second-order Taylor expansion around $(\bar{x},\bar{y},\bar{z})$**

$$f(x,y,z) \approx f(\bar{x},\bar{y},\bar{z}) + f_x(\bar{x},\bar{y},\bar{z})(x-\bar{x}) + f_y(\bar{x},\bar{y},\bar{z})(y-\bar{y}) + f_z(\bar{x},\bar{y},\bar{z})(z-\bar{z})$$
$$+\frac{1}{2} f_{xx}(\bar{x},\bar{y},\bar{z})(x-\bar{x})^2 + \frac{1}{2} f_{yy}(\bar{x},\bar{y},\bar{z})(y-\bar{y})^2 + \frac{1}{2} f_{zz}(\bar{x},\bar{y},\bar{z})(z-\bar{z})^2$$
$$+\frac{1}{2} f_{xy}(\bar{x},\bar{y},\bar{z})(x-\bar{x})(y-\bar{y}) + \frac{1}{2} f_{yx}(\bar{x},\bar{y},\bar{z})(y-\bar{y})(x-\bar{x}) + \ldots.$$

(Illustrative example in scalars)

- **Why a second-order approximation?**
  - **To capture (some…) effect of standard deviations of exogenous shocks on decision functions**
  - **e.g., σ_z**

$$\ln z_{t+1} = (1-\rho_z)\ln \bar{z} + \rho_z \ln z_t + \varepsilon_{t+1}$$

$$\varepsilon_{t+1} \sim \text{iid } N(0,\sigma_z^2)$$

  - **Recall first-order approximation displays certainty equivalence**

Linearization of the RBC Model

Define co-state vector and state vector

$$y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix} \qquad x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}$$

Order model's dynamic equations in a vector $\equiv f(y_{t+1}, y_t, x_{t+1}, x_t) = 0$

Consumption-leisure efficiency condition

Consumption-investment efficiency condition

Aggregate resource constraint

Law of motion for TFP

Linearization of the RBC Model

- A quadratic equation in the elements of g_x and h_x evaluated at the steady state

$$F_x(\overline{x},0) = f_{y_{t+1}}(\overline{x},0)\cdot g_x(\overline{x},0)\cdot h_x(\overline{x},0) + f_{y_t}(\overline{x},0)\cdot g_x(\overline{x},0) + f_{x_{t+1}}(\overline{x},0)\cdot h_x(\overline{x},0) + f_{x_t}(\overline{x},0) = 0$$

- Solve numerically for the elements of g_x and h_x (use fsolve OR conduct an eigenvalue decomposition in Matlab)
- Recall conjectured equilibrium decision rules

$$y_t = g(x_t,\sigma)$$

$$x_{t+1} = h(x_t,\sigma) + \eta\sigma\varepsilon_{t+1}$$

- First-order approximation is

$$y_t = g(x_t,\sigma) \approx g(\overline{x},0) + g_x(\overline{x},0)(x_t - \overline{x}) + \overbrace{g_\sigma}^{=0}(\overline{x},0)\sigma$$

$$x_{t+1} = h(x_t,\sigma) \approx h(\overline{x},0) + h_x(\overline{x},0)(x_t - \overline{x}) + \underbrace{h_\sigma}_{=0}(\overline{x},0)\sigma$$

SGU Theorem 1:

$g_\sigma = 0$ and $h_\sigma = 0$

SECOND-ORDER

- How?...

SECOND-ORDER

- How?...
- The model's dynamic expectational equations

$$
\begin{aligned}
E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t)\right] = 0 \\
&= E_t\left[f(g(x_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, x_t\right] \\
&= E_t\left[f(g(h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, \sigma), g(x_t, \sigma), h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, x_t\right] \\
&\equiv F(x_t, \sigma)
\end{aligned}
$$

$$\Downarrow$$

$$F_x(x_t, \sigma) = 0 \qquad F_\sigma(x_t, \sigma) = 0$$

SECOND-ORDER

- How?...
- The model's dynamic expectational equations

$$
\begin{aligned}
E_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t)\right] &= 0 \\
&= E_t\left[f(g(x_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1}, x_t\right] \\
&= E_t\left[f(g(h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1},\sigma), g(x_t,\sigma), h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1}, x_t\right] \\
&\equiv F(x_t,\sigma)
\end{aligned}
$$

$$\Downarrow$$

$$F_x(x_t,\sigma)=0 \qquad F_\sigma(x_t,\sigma)=0$$

$$F_{xx}(x_t,\sigma)=0 \qquad F_{\sigma\sigma}(x_t,\sigma)=0 \qquad F_{x\sigma}(x_t,\sigma)=0 \qquad F_{\sigma x}(x_t,\sigma)=0$$

SECOND-ORDER

- How?...need to compute the second derivatives of the equilibrium conditions
- SGU (2004 *JEDC*, p. 762)

$$
\begin{aligned}
[F_{xx}(\bar{x},0)]^i_{jk} = {} & ([f_{y'y'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{y'y}]^i_{\alpha\gamma}[g_x]^\gamma_k \\
& + [f_{y'x'}]^i_{\alpha\delta}[h_x]^\delta_k + [f_{y'x}]^i_{\alpha k})[g_x]^\alpha_\beta[h_x]^\beta_j \\
& + [f_{y'}]^i_\alpha[g_{xx}]^\alpha_{\beta\delta}[h_x]^\delta_k[h_x]^\beta_j \\
& + [f_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_{xx}]^\beta_{jk} \\
& + ([f_{yy'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{yy}]^i_{\alpha\gamma}[g_x]^\gamma_k + [f_{yx'}]^i_{\alpha\delta}[h_x]^\delta_k + [f_{yx}]^i_{\alpha k})[g_x]^\alpha_j \\
& + [f_y]^i_\alpha[g_{xx}]^\alpha_{jk} \\
& + ([f_{x'y'}]^i_{\beta\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{x'y}]^i_{\beta\gamma}[g_x]^\gamma_k + [f_{x'x'}]^i_{\beta\delta}[h_x]^\delta_k + [f_{x'x}]^i_{\beta k})[h_x]^\beta_j \\
& + [f_{x'}]^i_\beta[h_{xx}]^\beta_{jk} \\
& + [f_{xy'}]^i_{j\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{xy}]^i_{j\gamma}[g_x]^\gamma_k + [f_{xx'}]^i_{j\delta}[h_x]^\delta_k + [f_{xx}]^i_{jk} \\
= {} & 0; \quad i = 1,\ldots,n, \quad j,k,\beta,\delta = 1,\ldots,n_x; \quad \alpha,\gamma = 1,\ldots,n_y.
\end{aligned}
$$

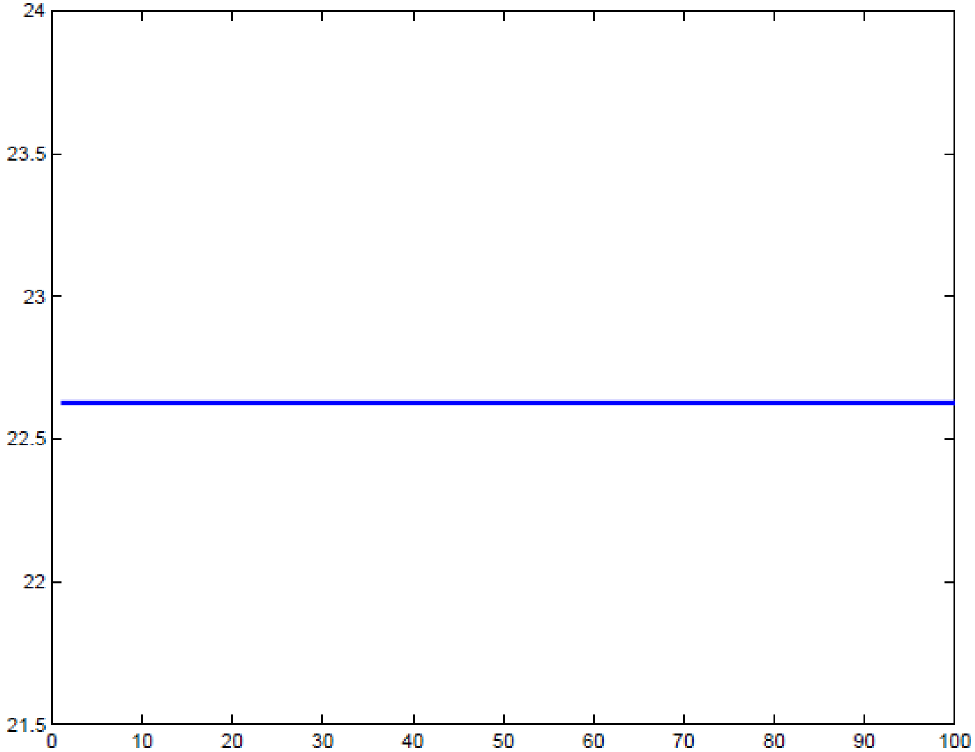
SECOND-ORDER

- How?...need to compute the second derivatives of the equilibrium conditions
- SGU (2004 *JEDC*, p. 762-763)

$$
\begin{aligned}
[F_{\sigma\sigma}(\bar{x},0)]^i &= [f_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_{\sigma\sigma}]^\beta \\
&+ [f_{y'y'}]^i_{\alpha\gamma} [g_x]^\gamma_\delta [\eta]^\delta_\xi [g_x]^\alpha_\beta [\eta]^\beta_\phi [I]^\phi_\xi \\
&+ [f_{y'x'}]^i_{\alpha\delta} [\eta]^\delta_\xi [g_x]^\alpha_\beta [\eta]^\beta_\phi [I]^\phi_\xi \\
&+ [f_{y'}]^i_\alpha [g_{xx}]^\alpha_{\beta\delta} [\eta]^\delta_\xi [\eta]^\beta_\phi [I]^\phi_\xi \\
&+ [f_{y'}]^i_\alpha [g_{\sigma\sigma}]^\alpha \\
&+ [f_y]^i_\alpha [g_{\sigma\sigma}]^\alpha \\
&+ [f_{x'y'}]^i_{\beta\gamma} [g_x]^\gamma_\delta [\eta]^\delta_\xi [\eta]^\beta_\phi [I]^\phi_\xi \\
&+ [f_{x'x'}]^i_{\beta\delta} [\eta]^\delta_\xi [\eta]^\beta_\phi [I]^\phi_\xi \\
&= 0; \quad i = 1,\ldots,n; \quad \alpha,\gamma = 1,\ldots,n_y; \quad \beta,\delta = 1,\ldots,n_x; \\
&\quad \phi,\xi = 1,\ldots,n_\varepsilon.
\end{aligned}
$$

SECOND-ORDER APPROXIMATION OF RBC MODEL

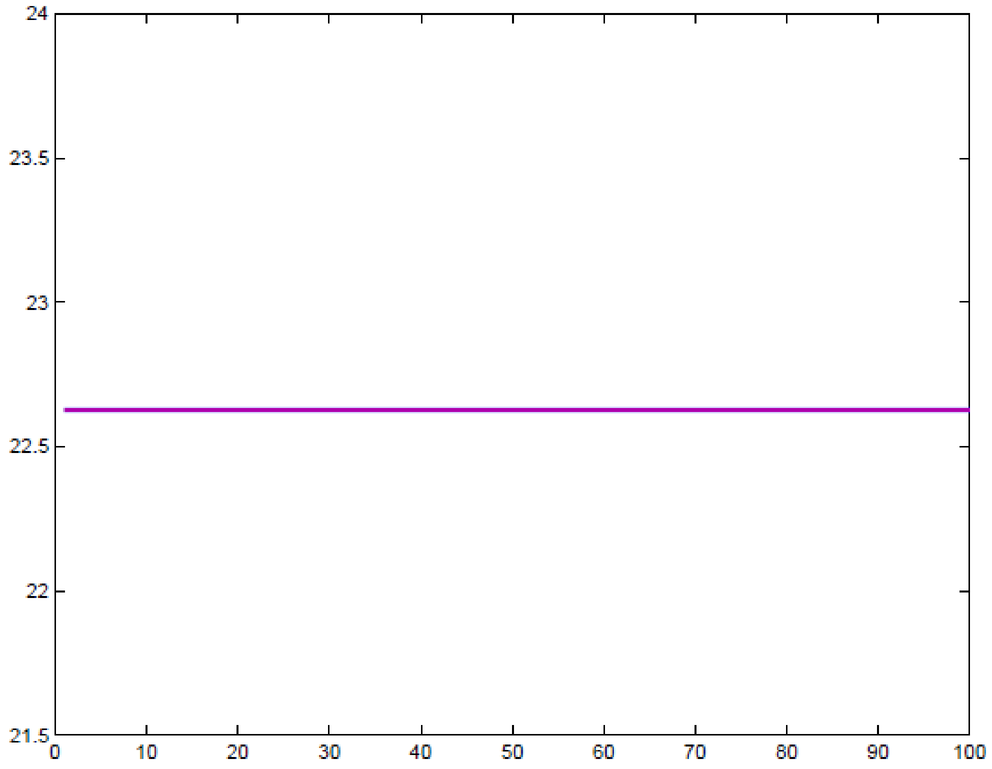
- Suppose sequence of realized shocks to TFP is always 0
- Compare linear- vs. second-order simulations
- For $\sigma_z = 0$



SECOND-ORDER APPROXIMATION OF RBC MODEL

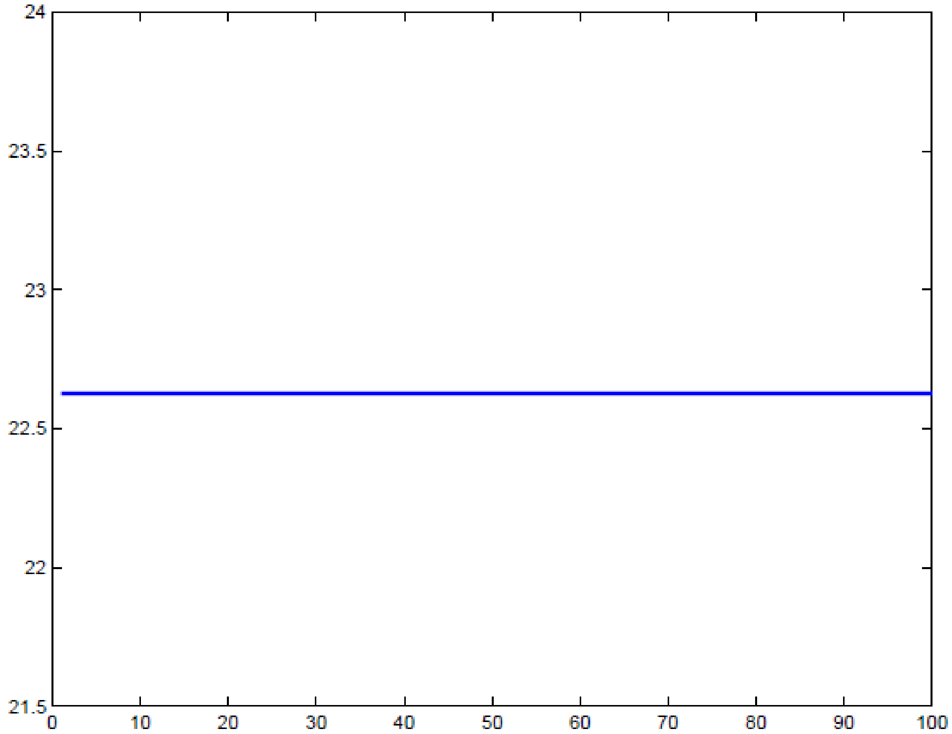
- **Suppose sequence of realized shocks to TFP is always 0**
- **Compare linear- vs. second-order simulations**
- **For $\sigma_z = 0$**

First-order and second-order simulations identical



SECOND-ORDER APPROXIMATION OF RBC MODEL

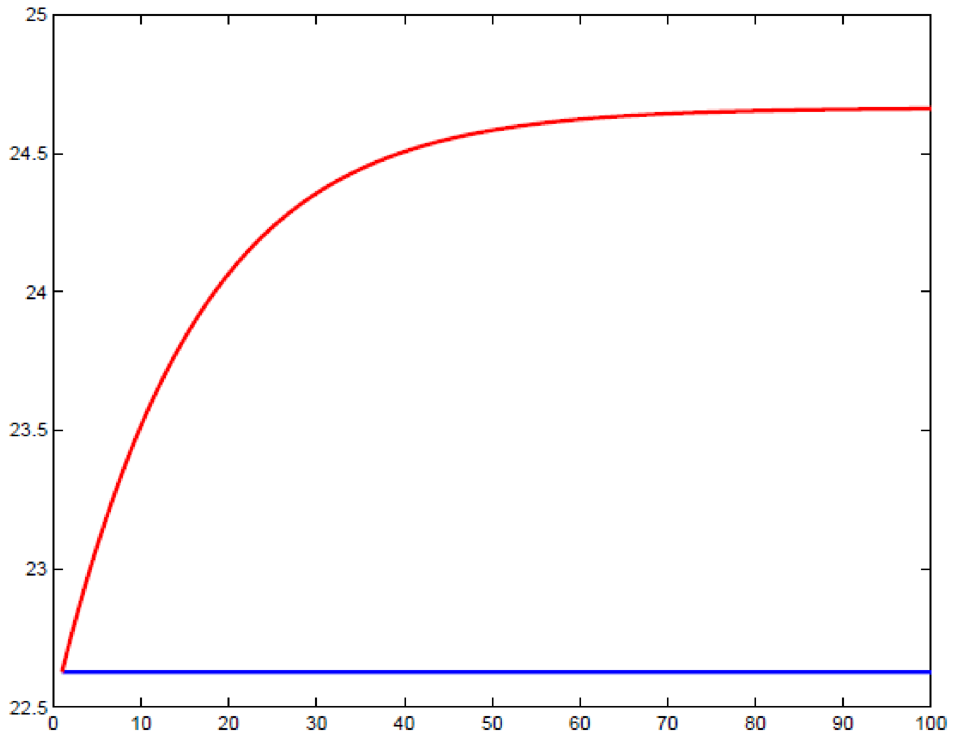
- **Suppose sequence of realized shocks to TFP is always 0**
- **Compare linear- vs. second-order simulations**
- **For σ_z = 0.007**



SECOND-ORDER APPROXIMATION OF RBC MODEL

- Suppose sequence of realized shocks to TFP is always 0
- Compare linear- vs. second-order simulations
- For $\sigma_z = 0.007$

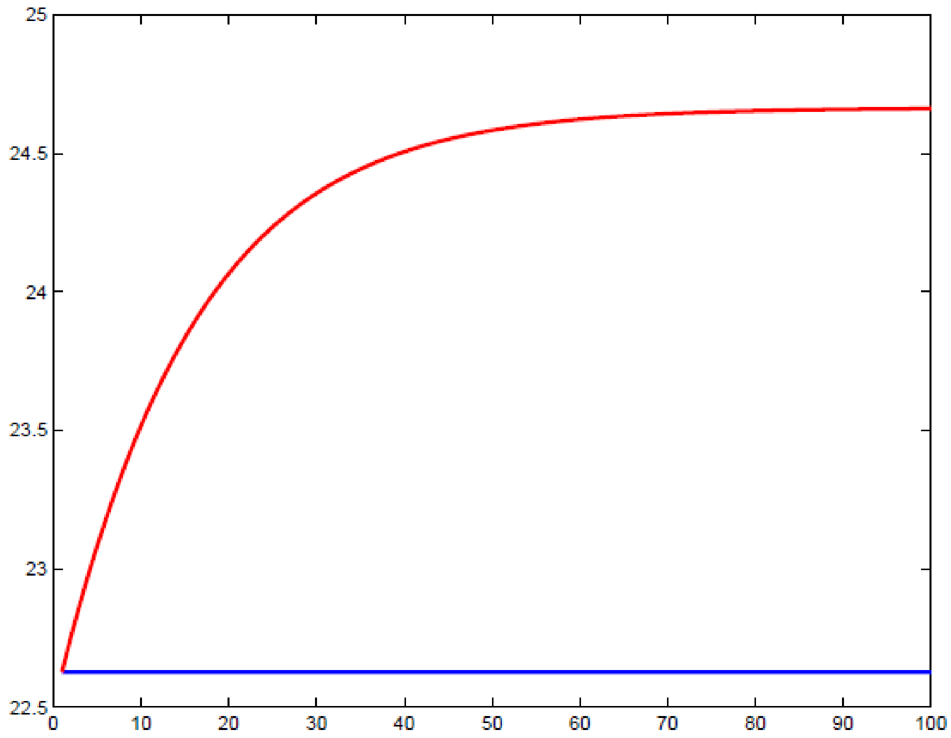
First-order and second-order simulations NOT identical



SECOND-ORDER APPROXIMATION OF RBC MODEL

- Suppose sequence of realized shocks to TFP is always 0
- Compare linear- vs. second-order simulations
- For σ_z = 0.007

First-order and second-order simulations NOT identical



"Stochastic" steady state

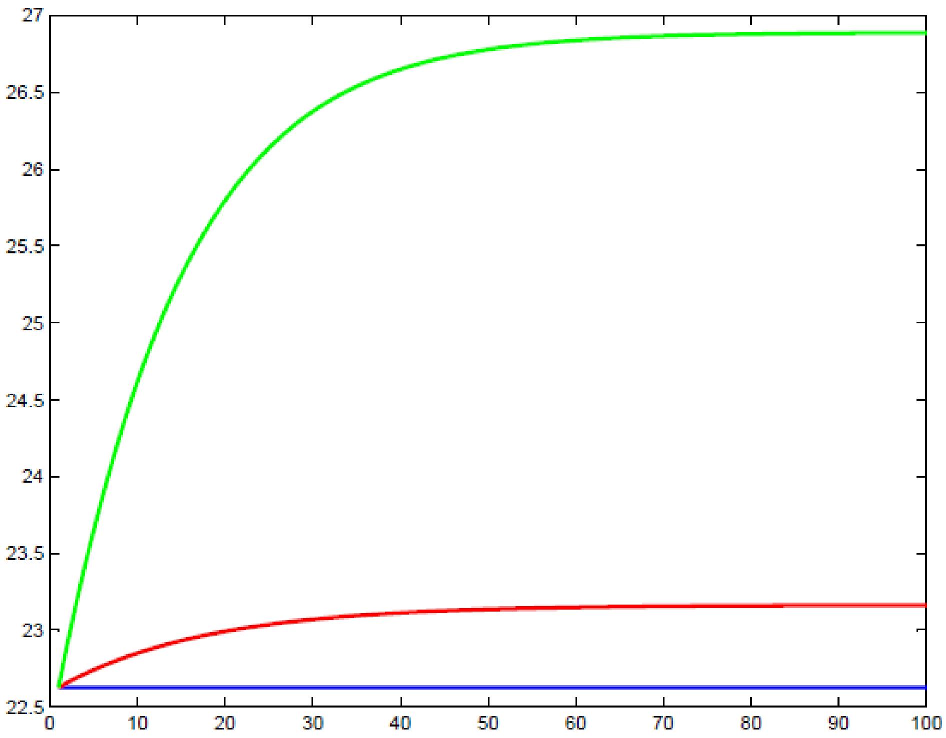
(mean of the ergodic distribution)

Deterministic steady state

SECOND-ORDER APPROXIMATION OF RBC MODEL

- Suppose sequence of realized shocks to TFP is always 0
- Compare linear- vs. second-order simulations
- Compare σ_z = 0.007 vs. σ_z = 0.021

First-order and second-order simulations NOT identical



"Stochastic" steady state

(mean of the ergodic distribution)

Deterministic steady state

Stochastic Steady State

- **Second-order approximation terms for the decision functions $g(x_t, \sigma)$ and $h(x_t, \sigma)$ start capturing the "precautionary savings" motive**
- **Precaution against what?**

Stochastic Steady State

- Second-order approximation terms for the decision functions $g(x_t, \sigma)$ and $h(x_t, \sigma)$ start capturing the "precautionary savings" motive
- Precaution against what?
- Against how large shocks to the exogenous processes (e.g., TFP) could be…
- …NOT the actual realizations of the TFP shocks
- $g_{\sigma\sigma}(x_t, \sigma)$ and $h_{\sigma\sigma}(x_t, \sigma)$ are NOT zero
- (Recall $g_{\sigma}(x_t, \sigma) = 0$ and $h_{\sigma}(x_t, \sigma) = 0$)

Stochastic Steady State

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Can we compute by hand ("fsolve") the stochastic steady state?

STOCHASTIC STEADY STATE

- **Shut down all shocks and set exogenous variables at their means**
- **The idea: let economy run for many (infinite) periods**
  - **Time eventually "doesn't matter" any more**
  - **Drop all time indices**

$$-\frac{u_n(\bar{c},\bar{n})}{u_c(\bar{c},\bar{n})} = \bar{z}F_n(\bar{k},\bar{n})$$

$$u_c(\bar{c},\bar{n}) = \beta u_c(\bar{c},\bar{n})\left[\bar{z}F_k(\bar{k},\bar{n}) + 1 - \delta\right]$$

$$\bar{c} + \delta\bar{k} = \bar{z}F(\bar{k},\bar{n})$$

$$\ln\bar{z} = (1-\rho_z)\ln\bar{z} + \rho_z \ln\bar{z} \Rightarrow \bar{z} = \bar{z} \quad \text{(a parameter of the model)}$$

- **Given functional forms and parameter values, solve for** $(\bar{c},\bar{n},\bar{k})$
  - **The steady state of the model**
  - **Taylor expansion around this point**

STOCHASTIC STEADY STATE

- ❒ **Second-order approximation terms for the decision functions $g(x_t, \sigma)$ and $h(x_t, \sigma)$ start capturing the "precautionary savings" motive**
- ❒ **Precaution against what?**
- ❒ **Against how large shocks to the exogenous processes (e.g., TFP) could be…**
- ❒ **…NOT the actual realizations of the TFP shocks**
- ❒ **$g_{\sigma\sigma}(x_t, \sigma)$ and $h_{\sigma\sigma}(x_t, \sigma)$ are NOT zero**
- ❒ **(Recall $g_{\sigma}(x_t, \sigma) = 0$ and $h_{\sigma}(x_t, \sigma) = 0$)**

Can we compute by hand ("fsolve") the stochastic steady state?

NO! – HAS TO BE NUMERICALLY COMPUTED