CONSUMPTION-SAVINGS MODEL

JANUARY 19, 2018

APPLICATIONS

- **Use (solution to) stochastic two-period model to illustrate some basic results and ideas in**
  - **Consumption research**
  - **Asset pricing research**

- **Certainty-equivalent consumption**
  - **Assuming**
    - **Quadratic period-utility** $u(c)=\gamma c-\frac{\alpha c^2}{2}$
    - **Risk-free asset returns**
    - **Risky period-2 income (with arbitrary distribution)**

- **Risk aversion**

- **Precautionary savings**

- **Appendix: Asset pricing**

CERTAINTY EQUIVALENCE

- **Assume quadratic utility**

$$v(c_1, c_2) = u(c_1) + u(c_2) = \gamma c_1 - \frac{\alpha c_1^2}{2} + \gamma c_2 - \frac{\alpha c_2^2}{2}$$

- **Assume interest rate is not state contingent**

$$r_1^H = \bar{r}_1 = r_1^L = r_1$$

risk-free interest rate

- **Insert in definition of solution to intertemporal problem**

$$c_2^H + \underbrace{a_2^H}_{= 0} = y_2^H + (1 + r_1)a_1 \qquad c_2^M + \underbrace{a_2^M}_{= 0} = \bar{y}_2 + (1 + r_1)a_1 \qquad c_2^L + \underbrace{a_2^L}_{= 0} = y_2^L + (1 + r_1)a_1 \qquad c_1 + a_1 = y_1 + (1 + r_0)a_0$$

Euler eqn the key

$$u'(c_1) = E_1\left[u'(c_2)(1 + r_1)\right]$$

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= 0 = 0 = 0

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$$u'(c_1)=E_1\left[u'(c_2)(1+r_1)\right] \longrightarrow \gamma-\alpha c_1=E_1\left[(\gamma-\alpha c_2)(1+r_1)\right]$$

$$\longrightarrow \gamma-\alpha c_1=q(\gamma-\alpha c_2^H)(1+r_1)+p(\gamma-\alpha c_2^M)(1+r_1)+(1-p-q)(\gamma-\alpha c_2^L)(1+r_1)$$

$$\longrightarrow \gamma-\alpha c_1=(1+r_1)\left[q(\gamma-\alpha c_2^H)+p(\gamma-\alpha c_2^M)+(1-p-q)(\gamma-\alpha c_2^L)\right]$$

$$=(1+r_1)\left[\gamma-\alpha\left(\underbrace{qc_2^H+pc_2^M+(1-p-q)c_2^L}_{=E_1c_2}\right)\right]$$

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$$u'(c_1)=E_1\left[u'(c_2)(1+r_1)\right]\longrightarrow\gamma-\alpha c_1=E_1\left[(\gamma-\alpha c_2)(1+r_1)\right]$$

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$$=(1+r_1)\left[\gamma-\alpha\underbrace{\left(qc_2^H+pc_2^M+(1-p-q)c_2^L\right)}\right]$$

$$\longrightarrow\quad\gamma-\alpha c_1=(1+r_1)\left[\gamma-\alpha E_1c_2\right]\longrightarrow c_1=-\frac{\gamma}{\alpha}r_1+(1+r_1)E_1c_2$$

CERTAINTY EQUIVALENCE

- **If not concerned with state-contingent solutions for c_2…**
- **…solution to consumer problem is an asset position and expected consumption profile $\left(c_1, E_1 c_2; a_1\right)$ that satisfies**
  - **Period-2 budget constraint in expectation**

$$E_1 c_2 = (1+r_1)a_1 + E_1 y_2$$

  - **Euler equation**

$$c_1 = -\frac{\gamma}{\alpha} r_1 + (1+r_1) E_1 c_2$$

  - **Period-1 budget constraint** $c_1 + a_1 = y_1 + (1+r_0)a_0$

taking as given $\left(r_1; y_1, a_0, r_0\right)$ and the stochastic distribution *G*(.) of y_2

- **Optimal period-1 consumption**

$$c_1 = \underbrace{-\frac{\gamma}{\alpha}\left(\frac{r_1}{1+(1+r_1)^2}\right)}_{\equiv A} + \underbrace{\left(\frac{(1+r_1)^2}{1+(1+r_1)^2}\right)}_{\equiv B}\left(y_1 + (1+r_0)a_0\right) + \underbrace{\left(\frac{1+r_1}{1+(1+r_1)^2}\right)}_{\equiv C} E_1 y_2$$

CERTAINTY EQUIVALENCE

- **Optimal period-1 (current) consumption**

$$c_1 = A + B\cdot\left(y_1 + (1+r_0)a_0\right) + C\cdot E_1 y_2$$

  - **Depends only on the mean of risky future income, $E_1 y_2$**
  - **Independent of second- and higher-moments of risky future income**

- **Distribution function *G*(.) of period-2 income**

$$y_2 = \begin{cases} y_2^{\,H} & \text{probability q} \\ \bar{y}_2 & \text{probability p} \\ y_2^{\,L} & \text{probability 1-p-q} \end{cases}$$

$$E_1 y_2 = \bar{y}_2$$

$$\text{Var}\, y_2 = q\left(y_2^H - \bar{y}_2\right)^2 + (1-p-q)\left(y_2^L - \bar{y}_2\right)^2$$

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- **Certainty Equivalence**
  - **Mean-preserving spreads of $G(.)$ do not affect optimal choice of c_1**
  - **E.g., (p = 1, q = 0)**
    - **Period-2 income has no risk**
    - **But c_1 is identical**
    - **s_1 (period-1 savings) is identical**

CERTAINTY EQUIVALENCE

- ❑ **A benchmark result in intertemporal consumption theory**

- ❑ **Result depends on**
  - ❑ **Quadratic utility**
  - ❑ **Riskless (aka non-state-contingent) asset returns**
  - ❑ **Only source of risk is income risk**

- ❑ **Strong implication: risk about future (income) does not affect current consumption and savings decisions**
  - ❑ **Intuitively plausible?**
  - ❑ **Empirically relevant?**
  - ❑ **Probably not…but why not?**

- ❑ **Model does feature both**
  - ❑ **Income risk (Var y_2 > 0)**
  - ❑ **Risk averse utility with respect to consumption – need to define formally**

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- Appendix: Asset pricing

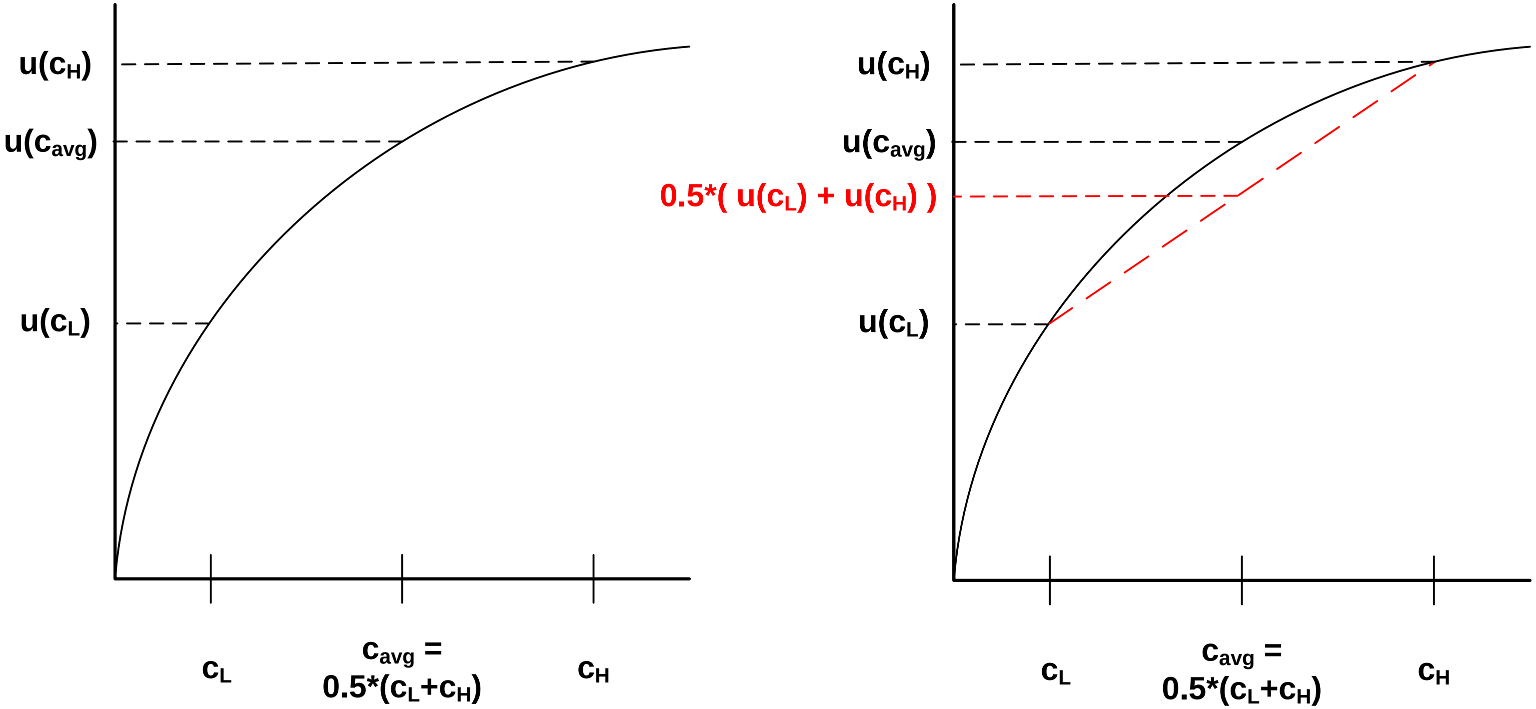
RISK AVERSION

- Illustrate with simple static example

- Utility function $u(c)$, with $u'(.) > 0$ and $u''(.) < 0$
- Two possible consumption outcomes
  - c^H with probability η
  - c^L with probability $1-\eta$

- Expected consumption is $\overline{c} = \eta c^H + (1-\eta)c^L$

RISK AVERSION



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- **Definition: an individual is risk averse (with respect to consumption risk) if**

$$u(\overline{c}) > \eta \cdot u\left(c^H\right) + (1-\eta) \cdot u\left(c^L\right)$$

JENSEN'S INEQUALITY

RISK AVERSION

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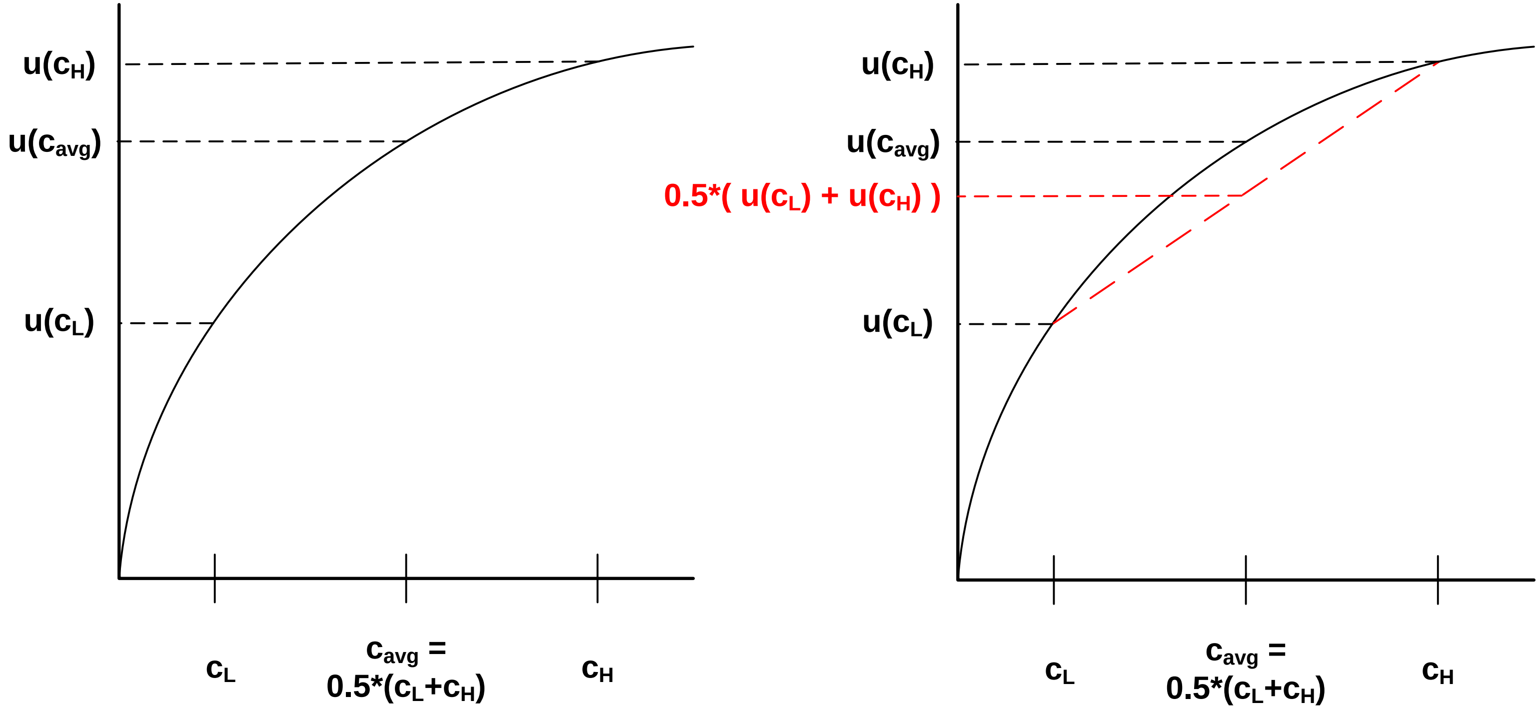
$$u(\overline{c}) > \eta \cdot u\left(c^H\right) + (1-\eta) \cdot u\left(c^L\right)$$ **JENSEN'S INEQUALITY**

- **Risk aversion**
  - **A preference for certain (deterministic) outcomes to risky (stochastic) outcomes**
  - **Embodied in strictly concave utility**

- **How to measure risk aversion?**
  - **Need to capture something about concavity of utility**

RISK AVERSION

- **How to measure?**



- **A candidate measure: $-u''(c)$**
  - **But not invariant to positive linear transformations of $u(.)$…**
  - **…even though implied choices are invariant to any monotonically increasing transformation of $u(.)$**

RISK AVERSION

- **Arrow-Pratt coefficient of absolute risk aversion (ARA)**

$$ARA(c) \equiv -\frac{u''(c)}{u'(c)}$$

Controls for linear transformations of *u*(.)

- **$ARA(c)$ gets at idea of risk aversion in level gains or losses of c from $E(c)$**
  - **Increasing ARA: $ARA'(c) > 0$**
  - **Decreasing ARA: $ARA'(c) < 0$**
    - **Most empirically-relevant case**
    - **Intuition – Richer people can afford to take a chance**

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- **Perhaps also useful to have measure of risk aversion in percentage gains or losses of *c* from *E*(*c*)**

- **Relative risk aversion (RRA)**

$$RRA(c) \equiv -\frac{cu''(c)}{u'(c)}\left(= c \cdot ARA(c)\right)$$

Adjusts for level of consumption/wealth

  - **Intuition – Controlling for income/consumption, richer people cannot afford to take a chance anymore than anyone else**

RISK AVERSION

- ☐ **CRRA**

$$v(c_1, c_2) = \underbrace{\frac{c_1^{1-\sigma} - 1}{1-\sigma}}_{\equiv u(c_1)} + \underbrace{\frac{c_2^{1-\sigma} - 1}{1-\sigma}}_{\equiv u(c_2)} \qquad \sigma > 0$$

Continuing to assume utility is additively-separable over time

- ☐ **Attitude of consumers toward smoothing consumption between time periods**
  - ☐ **IES = 1/*σ***

- ☐ **Attitude of consumers toward risky outcomes within a given time period**

$$RRA(c) = -\frac{cu''(c)}{u'(c)} = \ldots \qquad ARA(c) = -\frac{u''(c)}{u'(c)} = \ldots$$

RISK AVERSION

- **CRRA**

$$v(c_1,c_2)=\underbrace{\frac{c_1^{1-\sigma}-1}{1-\sigma}}_{\equiv u(c_1)}+\underbrace{\frac{c_2^{1-\sigma}-1}{1-\sigma}}_{\equiv u(c_2)} \qquad \sigma>0$$

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$$RRA(c)=-\frac{cu''(c)}{u'(c)}=\ldots \qquad ARA(c)=-\frac{u''(c)}{u'(c)}=\ldots$$

- **CRRA utility: σ governs both intertemporal attitudes and intratemporal (relative) risk attitudes!**
  - **Inverses of each other!**

- **Must/should IES and RRA be so directly related in reality?**
  - **Not at all…Epstein-Zin (EZ) utility function disentangles the two concepts**

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- **Precautionary savings**

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PRECAUTIONARY SAVINGS

- Certainty-equivalent consumption
  - Current consumption depends only on the mean of future risky income
  - Most important assumption: quadratic utility
  - Other necessary assumptions
    - Non-state-contingent asset returns
    - Future income the only source of risk

PRECAUTIONARY SAVINGS

- **Certainty-equivalent consumption**
  - **Current consumption depends only on the mean of future risky income**
  - **Most important assumption: quadratic utility**
- **Risk aversion (within period) with** $v(c_1,c_2)=u(c_1)+u(c_2)=\gamma c_1-\frac{\alpha c_1^2}{2}+\gamma c_2-\frac{\alpha c_2^2}{2}$ **?**
  - **Obviously $\neq$ 0! (whether RRA or ARA)**
  - **So why certainty equivalence?**

    i.e., why does future income risk "not matter" for current choices?

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$$\longrightarrow\quad\gamma-\alpha c_1=(1+r_1)\left[q(\gamma-\alpha c_2^H)+p(\gamma-\alpha c_2^M)+(1-p-q)(\gamma-\alpha c_2^L)\right]$$
$$=(1+r_1)\left[\gamma-\alpha\underbrace{\left(qc_2^H+pc_2^M+(1-p-q)c_2^L\right)}\right]$$

Because of linear *marginal* utility!

$$\longrightarrow\quad\gamma-\alpha c_1=(1+r_1)\left[\gamma-\alpha E_1c_2\right]\longrightarrow c_1=-\frac{\gamma}{\alpha}r_1+(1+r_1)E_1c_2$$

PRECAUTIONARY SAVINGS

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- **Marginal utility function of order one (or lower) *implies* risky future income doesn't matter for current consumption**

↓ **Contrapositve**

- **Risky future income matters for current consumption *implies* marginal utility function must be strictly convex**

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- **$u'''(c) > 0$ necessary for breaking certainty-equivalence result**
  - **(Given $u'(.) > 0$ and $u''(.) < 0$)**
  - **$u'''(.) > 0$ → $u''(.)$ increasing in c → $u'(.)$ decreasing less quickly as c↑**
  - **Not satisfied by quadratic utility**

PRECAUTIONARY SAVINGS

- ☐ **Assume utility with $u'''(c) > 0$**

$$v(c_1, c_2) = u(c_1) + u(c_2)$$

- ☐ **Assume interest rate is not state contingent**

$$r_1^H = \bar{r}_1 = r_1^L = r_1$$

risk-free interest rate

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risk-free interest rate

- **Insert in definition of solution to intertemporal problem**

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Euler eqn the key

$$u'(c_1) = E_1\left[u'(c_2)(1+r_1)\right] \longrightarrow u'(c_1) = (1+r_1)E_1\left[u'(c_2)\right]$$

$$\longrightarrow \quad u'(c_1) = (1+r_1)\left[\underbrace{qu'(c_2^H) + pu'(c_2^M) + (1-p-q)u'(c_2^L)}\right]$$

$\neq E_1 c_2$, so none of the subsequent steps with quadratic u(.) follow

- **$u'''(c) > 0$ → current consumption depends on distribution G(.) of future risk**
  - **i.e., on first- and (in principle) all higher-order moments of G(.)**

PRECAUTIONARY SAVINGS

- ❑ $u'''(c) > 0$ → current consumption depends on distribution $G(.)$ of future risk
- ❑ Optimal c_1 is smaller than certainty-equivalent c_1
  - ❑ Proof:

PRECAUTIONARY SAVINGS

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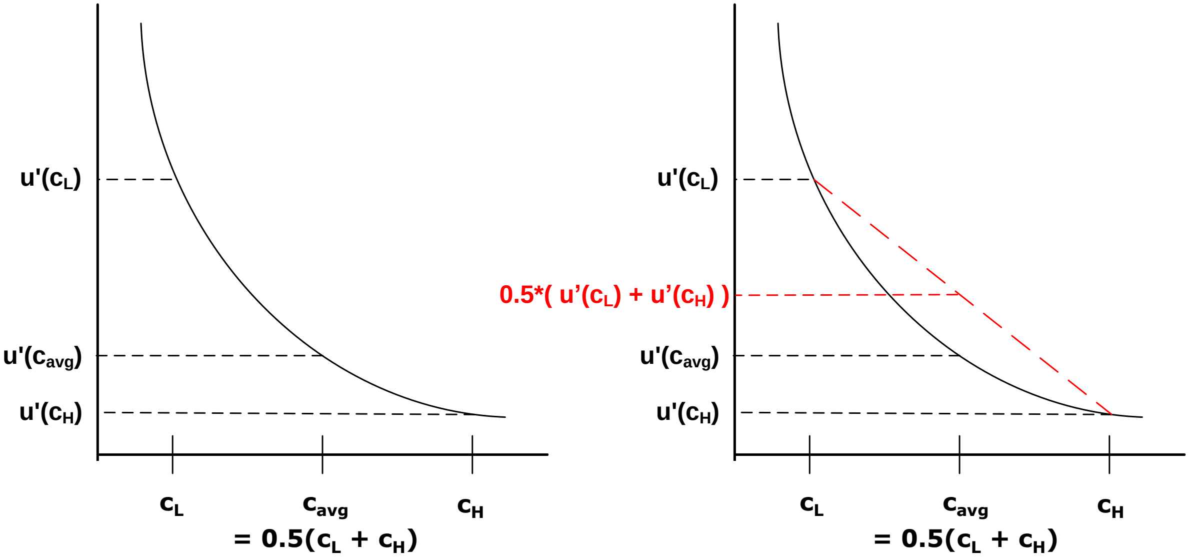
- ❑ Optimal c_1 is smaller than certainty-equivalent c_1
  - ❑ Proof:
- ❑ Implication: optimal s_1 is larger than certainty-equivalent s_1
- ❑ Precautionary Savings
  - ❑ Risk about the future induces prudent (cautious) choices in the present
  - ❑ Desire to build up a buffer stock of assets to ensure *c* does not fall too low in future
  - ❑ Risk aversion a necessary, but not sufficient, feature of preferences
  - ❑ Strictly convex marginal utility the key feature of preferences
  - ❑ Classic papers: Kimball (1990 *Econometrica*), Sandmo (1970 *Review of Economic Studies*)
- ❑ How to measure precautionary savings motive?
  - ❑ Need to capture something about convexity of marginal utility
  - ❑ Kimball (1990) provides clever insight

PRECAUTIONARY SAVINGS

- **How to measure?**



- **A candidate measure: *u'''*(*c*)**
  - **Analogy with measures of risk aversion**

PRECAUTIONARY SAVINGS

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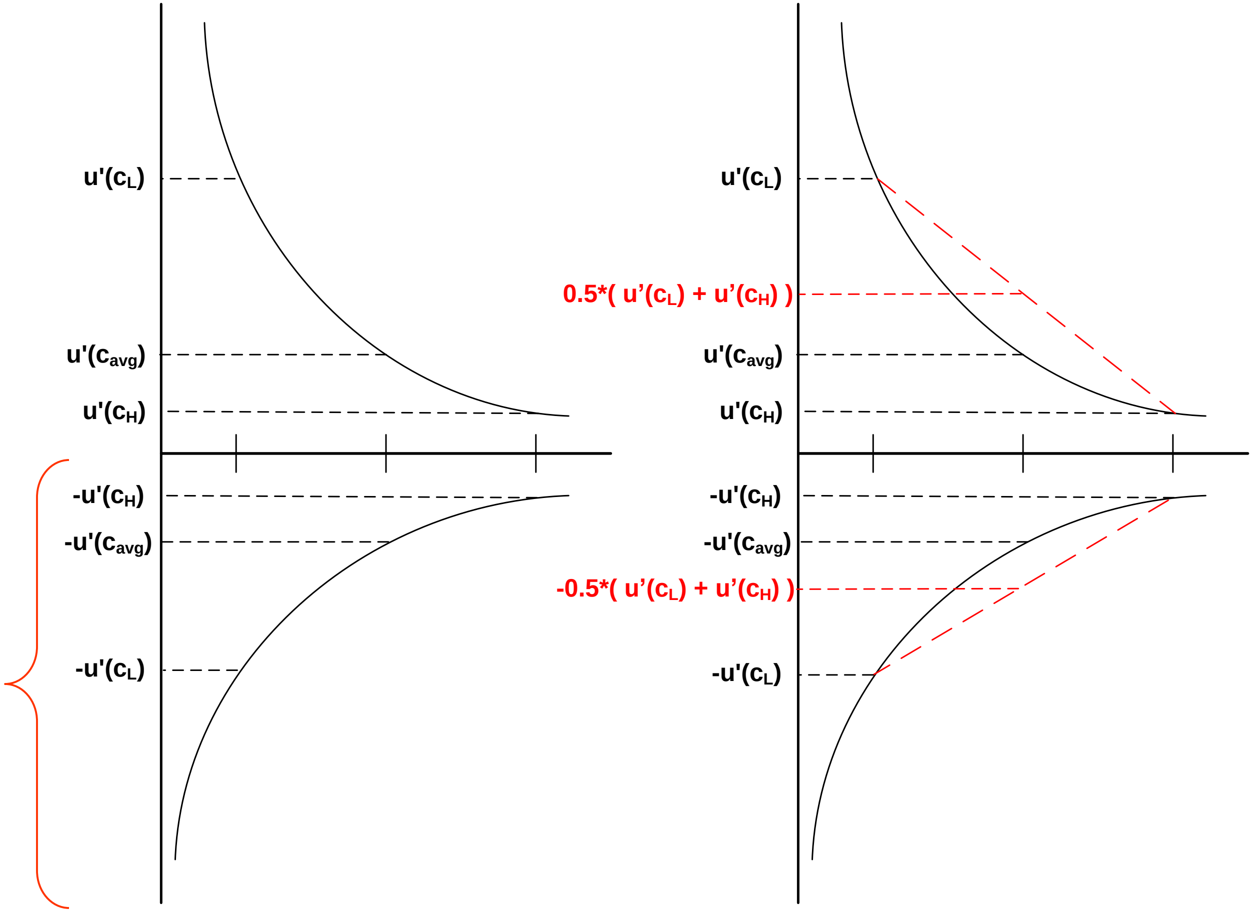

Kimball (1990):

Define $v(c) = -u'(c)$.

Then can apply standard theory of risk aversion to $v(c)$!

PRECAUTIONARY SAVINGS

- Coefficient of absolute prudence: $-\frac{u'''(c)}{u''(c)}$
- Coefficient of relative prudence: $-\frac{cu'''(c)}{u''(c)}$
- Measures of the sensitivity of optimal choice to risk
  - Governed by marginal utility function
- ARA and RRA measure the sensitivity of welfare to risk
  - Governed by the utility function

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- **ARA and RRA measure the sensitivity of welfare to risk**
  - **Governed by the utility function**
- **CRRA utility**

$$u(c)=\frac{c^{1-\sigma}-1}{1-\sigma}$$

$$-\frac{u'''(c)}{u''(c)}=\frac{\sigma+1}{c}$$

Absolute prudence

$$-\frac{cu'''(c)}{u''(c)}=\sigma+1$$

Relative prudence

  - **Displays constant relative prudence**
  - **Displays constant relative risk aversion**

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APPENDIX: ASSET PRICING

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ASSET MARKETS

- ❑ **Risk about the future (period 2) requires adopting a view about the nature of asset markets**

- ❑ **Continue with example of risky period-2 income**

$$y_2 = \begin{cases} y_2^H & \text{probability q} \\ \bar{y}_2 & \text{probability p} \\ y_2^L & \text{probability 1-p-q} \end{cases}$$

- ❑ **But now three "distinct" assets available for purchase in period 1**
  - ❑ **Asset a^H: purchase price R^H in period 1, pays off one unit in period 2 if y_2^H, zero else**
  - ❑ **Asset a^M: purchase price $\bar{R}$ in period 1, pays off one unit in period 2 if $\bar{y}_2$, zero else**
  - ❑ **Asset a^L: purchase price R^L in period 1, pays off one unit in period 2 if y_2^L, zero else**

- ❑ **Arrow-Debreu securities, aka contingent claims**

- ❑ **NOT EQUIVALENT to state-contingent asset returns on a *single* asset**

STATE-CONTINGENT CHOICES

- **Consumer problem**

$$\max u(c_1)+qu(c_2^H)+pu(c_2^M)+(1-p-q)u(c_2^L)+\lambda_1\left[y_1+a_0-c_1-R^H a_1^H-\bar{R}a_1^M-R^L a_1^L\right]$$
$$+q\lambda_2^H\left[y_2^H+a_1^H-c_2^H\right]+p\lambda_2^M\left[\bar{y}_2+a_1^M-c_2^M\right]+(1-p-q)\lambda_2^L\left[y_2^L+a_1^L-c_2^L\right]$$

- **FOCs**

BASICS OF ASSET PRICING

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- **FOCs**

- **Asset prices**

$$R^H=\frac{q\lambda_2^H}{\lambda_1}=\frac{qu'(c_2^H)}{u'(c_1)} \qquad \bar{R}=\frac{p\lambda_2^M}{\lambda_1}=\frac{pu'(c_2^M)}{u'(c_1)} \qquad R^L=\frac{(1-p-q)\lambda_2^L}{\lambda_1}=\frac{(1-p-q)u'(c_2^L)}{u'(c_1)}$$

  - $u'(c_2^j)/u'(c_1)$ **is willingness to intertemporally substitute consumption between period 1 and state *j* in period 2 – intertemporal MRS (IMRS)**

- **Contingent claims prices (aka Arrow-Debreu prices, aka state prices) reflect IMRS (if markets functioning well)**

- **In principle, allow for inferences about**
  - **Risk aversion**
  - **Prudence**
  - **Market participants' assessment of probabilities of event *j* occurring**

But which asset prices to empirically identify as which state prices?…

BASICS OF ASSET PRICING

- **Generalize the period-2 risk structure**
    - ***S*: number of possible realizations of y_2 (in richer models, risk in other primitives)**
    - ***R^j*: period-1 price of AD security that pays off one unit in state *j*, zero otherwise**
    - ***p^j*: probability of state *j* occurring in period 2, with** $\sum_{j=1}^{S} p^j = 1$
- **Lifetime expected utility** $u(c_1) + E_1 u(c_2) = u(c_1) + \sum_{j=1}^{S} p^j u(c_2^j)$
- **Period-1 budget constraint** $c_1 + \sum_{j=1}^{S} R^j a_1^j = y_1 + a_0$
- **State-*j* period-2 budget constraint** $c_2^j + \underbrace{a_2^j}_{= 0} = y_2^j + a_1^j, \quad j \in \{1, 2, 3, ..., S\}$

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- **AD price for state *j* (compute FOCs)**

$$R^j = \frac{p^j \lambda_2^j}{\lambda_1} = \frac{p^j u'(c_2^j)}{u'(c_1)}$$

- **Define** $R^f \equiv \sum_{j=1}^{S} R^j = \sum_{j=1}^{S} p^j \frac{u'(c_2^j)}{u'(c_1)} = E_1\left[\frac{u'(c_2)}{u'(c_1)}\right]$
    - **Is the price of a one-period riskless bond**

BASICS OF ASSET PRICING

- **One-period riskless bond**
  - **Purchase price R^f in period 1**
  - **Pays off one unit ("face value") in all states of the world in period 2**
  - **(Can scale to any arbitrary face value: $100 bonds, $1000 bonds, etc.)**

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- **Introduce in model**
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b_1: bond holdings carried from period 1 to period 2

- **FOC on b_1**

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$$R^f = E_1\left[\frac{\lambda_2}{\lambda_1}\right] = E_1\left[\frac{u'(c_2)}{u'(c_1)}\right]$$

Price of riskless bond reflects expected IMRS…

$$= \sum_{j=1}^{S} p^j \frac{u'(c_2^j)}{u'(c_1)} = \sum_{j=1}^{S} R^j$$

…and by no-arbitrage equals sum of state prices.

- **Result: risk-free bond price can be decomposed into state prices**
  - **A complete set of AD securities spans the risk space…**
  - **…which makes b_1 a redundant asset; consumer can synthesize b_1 himself**

- **Any asset can be decomposed into state prices (Cochrane, Chapter 3.1)**

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- **How do these asset structures affect consumer's intertemporal life?**

CONSUMPTION, SAVINGS, AND ASSET PRICES

- ❑ **Consumption smoothing** a primitive feature of preferences (*u'*(.)>0, *u''*(.)<0)
- ❑ Nature of asset markets affects ability to achieve consumption smoothing

- ❑ Two dimensions of consumption smoothing

- ❑ **Intertemporal consumption smoothing:** concavity of *u*(.) implies preference for ***low time-series-variance*** of consumption

$$R^f = E_1\left[\frac{u'(c_2)}{u'(c_1)}\right]$$

Expected IMRS = price of risk-free bond

$$\Leftrightarrow R^f u'(c_1) = E_1 u'(c_2)$$

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- Intratemporal consumption smoothing: concavity of *u*(.) implies preference for *low cross-state variance* of consumption *within any period that has risk*

$$R^j = \frac{p^j \lambda_2^j}{\lambda_1} = \frac{p^j u'(c_2^j)}{u'(c_1)}$$

- A high state price R^j reflects
  - High probability of state *j*
  - High *u*' (.) in state *j* – i.e., low consumption in state *j*
  - Or both

- View as intratemporal optimality condition across future state-contingent *c*

$$\frac{R^j / p^j}{R^k / p^k} = \frac{u'(c_2^j)}{u'(c_2^k)}, \qquad \forall j,k \in \{1,2,3,...,S\}$$

MRS across states *j*, *k* = (risk-adjusted) relative state price

CONSUMPTION, SAVINGS, AND ASSET PRICES

- ☐ **Define $m^j = R^j / p^j$ as discount factor for state j**
- ☐ **Intratemporal optimality condition**

$$\frac{m^j}{m^k} = \frac{u'(c_2^j)}{u'(c_2^k)}, \qquad \forall j, k \in \{1, 2, 3, ..., S\}$$

- ☐ **Intertemporal optimality between period 1 and state j in period 2**

$$m^j = \frac{u'(c_2^j)}{u'(c_1)}, \qquad \forall j \in \{1, 2, 3, ...S\}$$

- ☐ **Expected IMRS between period 1 and period 2**

Terminology:

Stochastic discount factor (SDF)

$$E_1 m \equiv \sum_{j=1}^{S} p^j m^j$$

$$= R^f = E_1\left[\frac{u'(c_2)}{u'(c_1)}\right]$$