Basics of Dynamic Stochastic (General) Equilibrium

HETEROGENEITY

- Implementing representative consumer
  - An infinity of consumers, each indexed by a point on the unit interval [0,1]
  - Each individual is identical in preferences and endowments
  - Implies aggregate consumption demand and asset demand

Aggregate consumption demand = One individual's consumption demand x **1**

Aggregate savings demand = One individual's savings demand x **1**

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- Under some particular types of heterogeneity, a representative-consumer foundation of aggregates exists
  - Provided complete set of Arrow-Debreu securities exists…
  - …to allow individuals to diversify away (insure) their idiosyncratic risk

- Consider heterogeneity
  - In income realizations (from Markov process)
  - In initial asset holdings *a*
  - In utility functions (application to CRRA utility)
  - Example: two types of individuals to illustrate

HETEROGENEITY

- **Two types of individuals, *i* ∈ {1,2}, each with population weight 0.5**

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u^i(c_t^i) \quad \textbf{subject to} \quad c_t^i + \sum_j R_t^j a_t^{ij} = y_t^i + a_{t-1}^i$$

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- Optimization between period *t* and state *j* in period *t*+1 (conditional on period *t* outcomes)

p^j denotes conditional probability of state *j* being realized in *t*+1

$$\frac{u_c^1(c_t^1)}{bu_c^1(c_{t+1}^{1j})} = \frac{p_{t+1}^j}{R_t^j} \qquad \frac{p_{t+1}^j}{R_t^j} = \frac{u_c^2(c_t^2)}{bu_c^2(c_{t+1}^{2j})}$$

Given all individuals base choices on same prices and probabilities

RISK SHARING

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In all states at all dates

- **PERFECT RISK SHARING**
  - **IMRS, for each state *j*, equated across individuals**
  - **Individuals experiencing idiosyncratic shocks can insure them away (provided complete markets)**

- **Risk sharing about equalizing fluctuations of *u*' (.) across individuals**
  - **Not about equalizing levels of *u*' (.) or consumption over time**

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- **If initial conditions, period-zero outcomes, and *u*(.) are identical (e.g., due to identical a_0 and realized y_0), then risk sharing → identical outcomes for all t**

→ A representative consumer

Risk Sharing

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Risk sharing across individuals ≈ consumption smoothing for a given individual

(= if initial conditions, *t*=0 outcomes, and *u*(.) identical)

RISK SHARING

- **Example: CRRA utility, but heterogenous RRA/IES**
  - **$\sigma^1 \neq \sigma^2$**

$$\left(\frac{c_t^1}{c_{t+1}^{1j}}\right)^{-\sigma^1} = \left(\frac{c_t^2}{c_{t+1}^{2j}}\right)^{-\sigma^2}$$

Perfect risk sharing

  - **IMRS equated across individuals**

- **Growth rates of consumption not equated unless $\sigma^1 = \sigma^2$**

$$\frac{c_{t+1}^{1j}}{c_t^1} = \left(\frac{c_{t+1}^{2j}}{c_t^2}\right)^{\sigma^2/\sigma^1}$$

AGGREGATION

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- **Allocations are Pareto-optimal (implied by First Welfare Theorem)**
  - **All MRS's (across individuals, states, and dates) are equated**
  - **Even though levels of consumption may differ across individuals**
  - **No individual can be made better off without making some agent worse off**
  - **(Pareto welfare concept takes distributions of outcomes as given)**
  - **Due to complete financial markets**

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- **Pareto-optimal allocations + heterogeneity of utility functions**
  - **There exists a utility function $u(c)$ in aggregate $c = c^1 + c^2$ that leads to the same aggregates (Constantanides (1982)); if CRRA, $u(.)$ has $\sigma \in (\sigma^1, \sigma^2)$**

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- **Pareto-optimal allocations + heterogeneity of utility functions**
  - **There exists a utility function $u(c)$ in aggregate $c = c^1 + c^2$ that leads to the same aggregates; proof relies on general equilibrium theory**

AGGREGATION

- Now consider **economy-wide aggregates**

$$c_t = 0.5c_t^1 + 0.5c_t^2$$

Aggregate consumption

$$y_t = 0.5y_t^1 + 0.5y_t^2$$

Aggregate income (endowment)

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(For each type of asset) $$a_t = 0.5a_t^1 + 0.5a_t^2$$ **Aggregate assets?**

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- In aggregate, must be some tangible asset(s) backing them (gen. equil.)

- No physical assets in model so far → a_t = ***0 in aggregate*** for all t !

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- Heterogeneous individuals creating/buying/selling assets vis-à-vis each other
- Richer models
  - Mediate through "banking" or "insurance" markets, etc.
  - But only meaningful if some friction/imperfections in model of financial markets…
  - …otherwise identical outcomes (in which case "banking" sector is a "veil")

AGGREGATION

- **Economy-wide aggregates**

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Asset market clearing condition
(for each type of asset)

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- **Aggregate savings = $a_t - a_{t-1}$ = 0 for all t**

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- **Aggregate together two types' budget constraints**

$$c_t^1 + \sum_j R_t^j a_t^{1j} = y_t^1 + a_{t-1}^1 \qquad c_t^2 + \sum_j R_t^j a_t^{2j} = y_t^2 + a_{t-1}^2$$

  - **Weight by share of population**
  - **Impose asset-market clearing condition(s)**

$$\Rightarrow \quad 0.5(c_t^1 + c_t^2) + \sum_j R_t^j 0.5\underbrace{(a_t^{1j} + a_t^{2j})}_{= 0 \text{ across } j} = 0.5(y_t^1 + y_t^2) + 0.5\underbrace{(a_{t-1}^1 + a_{t-1}^2)}_{= 0}$$

AGGREGATION

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$$\Rightarrow \; c_t = y_t$$

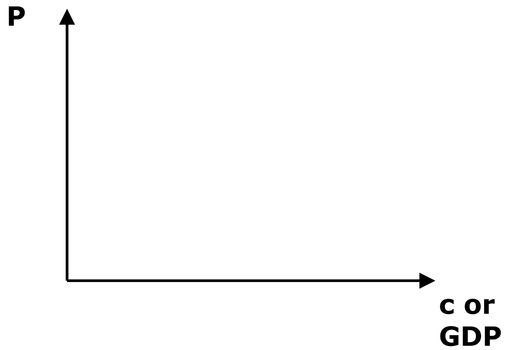
Goods market clearing condition – aka resource constraint

A general procedure for constructing economy-wide resource constraint

goods available = goods used

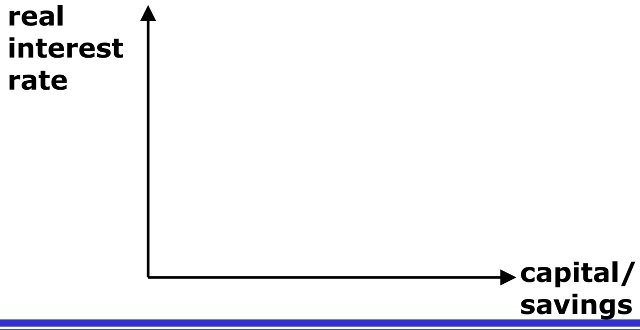
THE THREE MACRO (AGGREGATE) MARKETS

- **Goods Markets**



- **Labor Markets**

- **Capital/Savings/Funds/Asset Markets (aka Financial Markets)**



TOWARDS DYNAMIC GENERAL EQUILIBRIUM

- Lifecycle/permanent income consumption model the most basic building block of all macro models
- Dynamic stochastic general equilibrium (DSGE) theory
  - (DGE if deterministic)
  - GE: simultaneous determination of prices and quantities in all markets (macro markets: goods, labor, capital)

- Foundations of baseline DSGE model
  - Representative consumer
  - Representative firm
  - Perfect competition in all markets
  - Rational expectations
  - Perfect AD financial markets

THE REAL BUSINESS CYCLE MODEL

Kydland and Prescott (1982), Long and Plosser (1983), King, Plosser, and Rebelo (1988)

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- Foundations of the RBC model
  - Without optimizing consumers: Solow growth model
  - With optimizing consumers: Ramsey/Cass/Koopmans model

"The Solow model"

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$$y_t + (1+r_t)a_{t-1}$$

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- **Now suppose y_t is labor income**

$$y_t = w_t n_t$$

- **Normalize "time available" in each time period to one unit**
  - **Individual decides how to divide between "labor" and "leisure"**
    - **(Basic models: leisure is all "non-labor," but empirical and theoretical work recently studying the importance of finer categorizations of "non-labor time" for macro issues – e.g., search and matching theory)**
  - **Labor = n_t ←→ leisure is l_t = 1-n_t**

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- **Assert that individuals care about leisure, $u(c_t, \ell_t)$**
  - **$u_{ct} > 0$, $u_{lt} > 0$, $u_{cct} < 0$, $u_{llt} < 0$**
  - **Inada conditions on both *c* and *l***

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- **Sometimes more convenient to represent as** $u(c_t, n_t)$
  - $u_{ct} > 0$, $u_{nt} < 0$, $u_{cct} < 0$, $u_{nnt} > 0$ **(strictly decreasing and convex in *n*)**

TOWARDS DYNAMIC GENERAL EQUILIBRIUM

- **Intertemporal optimization problem**

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, n_t^S) \quad \textbf{subject to} \quad c_t + a_t = w_t n_t^S + (1+r_t) a_{t-1}$$

  - **Individual takes as given $\{w_t, r_t\}_{t=0,1,2,\ldots}$ -- price-taker in labor market**
    - **From perspective of individual, (w,r) evolve as Markov**
  - **Notation *n^S* emphasizes individual's supply of labor**

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- **Recursive representation**
  - **State vector in arbitrary period *t*: [ ]**

$$V(\qquad) =$$

Towards Dynamic General Equilibrium

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  - State vector in arbitrary period *t*: $[a_{t-1}; w_t, r_t]$

  $$V(a_{t-1}; w_t, r_t) = \max_{\{c_t, n_t^S, a_t\}} \left\{ u(c_t, n_t^s) + \beta E_t V(a_t; w_{t+1}, r_{t+1}) \right\}$$

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- **FOCs**

c_t :

n^S_t :

LABOR SUPPLY

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$$\left. \begin{aligned} \boldsymbol{c_t:} \quad & u_{ct} - \lambda_t = 0 \\ \boldsymbol{n^S_t:} \quad & u_{nt} + \lambda_t w_t = 0 \end{aligned} \right\} \quad -\frac{u_n(c_t, n_t^S)}{u_c(c_t, n_t^S)} = w_t$$

CONSUMPTION-LEISURE OPTIMALITY CONDITION

A *static* condition

Labor Supply

$$-\frac{u_n(c_t, n_t^S)}{u_c(c_t, n_t^S)} = w_t \qquad \Rightarrow \quad n_t^S = n^S(w_t; c_t)$$

- **Consumption-leisure (aka consumption-labor) optimality condition**
  - **An intratemporal optimality condition**

- **Defines period-*t* labor supply function**
  - **For given individual…**
  - **…but if representative agent, equivalent to aggregate labor supply**
  - **Note: for given *c***

LABOR SUPPLY

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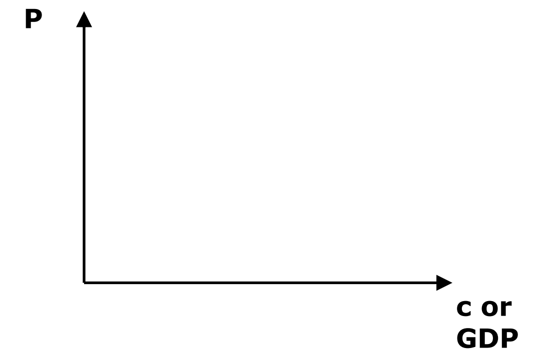
  - **Note: for given *c***

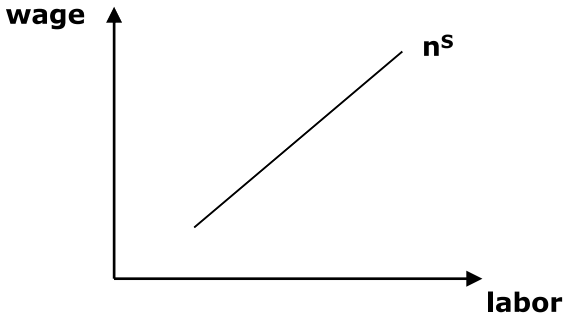
- **Example:** $u(c,n) = \ln c - \dfrac{\theta}{1+1/\psi} n^{1+1/\psi}$
  - **Compute labor supply function?**
  - **Compute elasticity of n^S_t with respect to w_t?**
    Frisch elasticity of labor supply

THE THREE MACRO (AGGREGATE) MARKETS

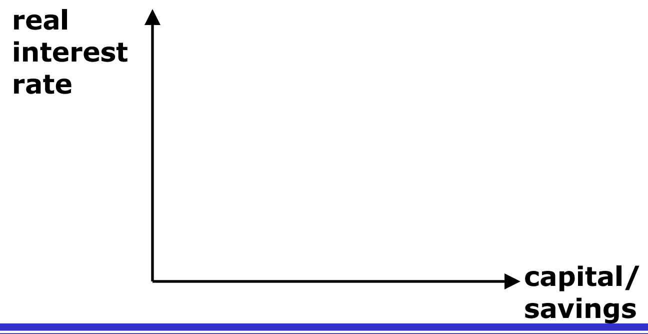
- **Goods Markets**



- **Labor Markets**



- **Capital/Savings/Funds/Asset Markets (aka Financial Markets)**



PRODUCTION OF GOODS

- **Representative firm produces the numeraire output good of the economy**
- **A homogenous output good**
- **Perfect competition in goods supply**
- **Inputs**
  - **Labor**
  - **Capital**
    - **E.g., machines, factories, computers, intangibles, …**

- **Firm produces using a (aggregate) production technology**

$$y_t = z_t \cdot f(k_t^D, n_t^D)$$

  - **k^D the firm's capital demand**
  - **n^D the firm's labor demand**
  - **$f(.)$ often assumed CRS (Cobb-Douglas, in particular)**
  - **z_t a process that shifts the production function**

- **Empirically identify z_t as Solow residual**
  - **Growth theory: z deterministic**
  - **Business cycle theory: z stochastic (Markov)**

PRODUCTION OF GOODS

- **Representative firm profit maximization**
  - **Price taker in capital market, labor market, and output market**
  - **Baseline model(s)**
    - **Firm hires/rents labor and capital each period**
    - **Firm does not “own” any capital or labor (without loss of generality if no financial market imperfections)**

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$$\max_{n_t^D, k_t^D} \left(z_t f(k_t^D, n_t^D) - w_t n_t^D - r_t^k k_t^D \right)$$

- **FOCs**

$\boldsymbol{n^D_t}$ **:** $z_t f_n(k_t^D, n_t^D) - w_t = 0$

$\boldsymbol{k^D_t}$ **:** $z_t f_k(k_t^D, n_t^D) - r_t^k = 0$

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- **FOCs**

n^D_t : $z_t f_n(k_t^D, n_t^D) - w_t = 0$ **DEFINES labor demand function $n^D(w_t)$**

k^D_t : $z_t f_k(k_t^D, n_t^D) - r_t^k = 0$ **DEFINES capital demand function $k^D(r^k_t)$**

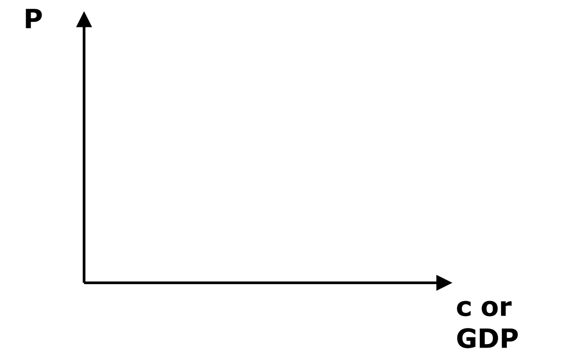
For a given firm

If rep. firm, equivalent to aggregate factor demands

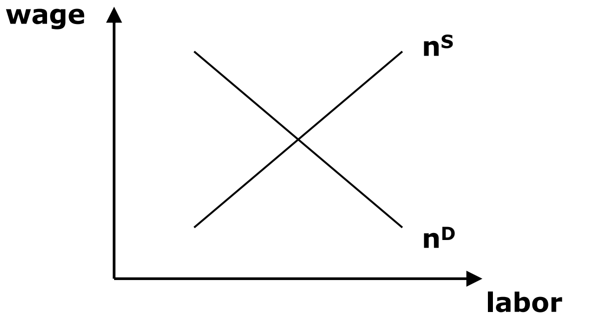
- **Firms entirely static entities in baseline macro model(s)**
  - **Contrast with consumers**
  - **(NK theory and matching theory: firms are dynamic entities)**

The Three Macro (Aggregate) Markets

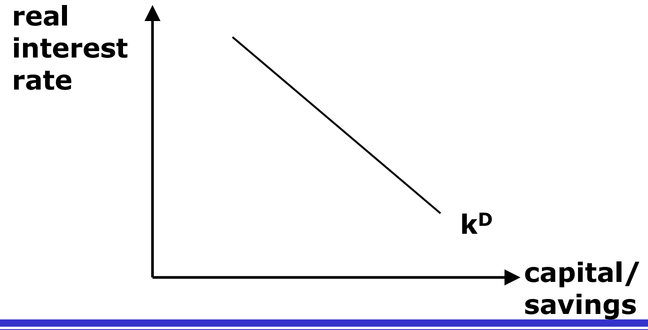
- Goods Markets



- Labor Markets



- Capital/Savings/Funds/Asset Markets (aka Financial Markets)



CAPITAL SUPPLY

- **Baseline model(s)**
  - **Physical capital takes "time to build"**
    - **Simplest: one-period lag between building and using capital**
  - **Closed economy**
    - **Aggregate capital demand must be supplied domestically**

- **Consumer intertemporal optimization problem**

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, n_t^S) \quad \textbf{subject to} \quad c_t + a_t = w_t n_t^S + (1+r_t) a_{t-1}$$

  - a_{t-1} **is a given individual's pre-determined stock of assets**
  - **Representative agent:** a_{t-1} **is economy's pre-determined stock of assets**

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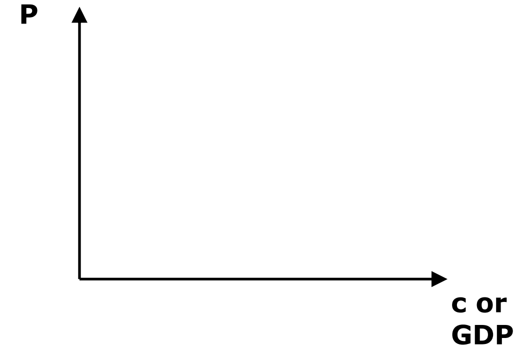
- **Capital-market clearing in each period *t***

$$k_t^D = a_{t-1} \left(= k_t^S \right)$$

THE THREE MACRO (AGGREGATE) MARKETS

- **Goods Markets**



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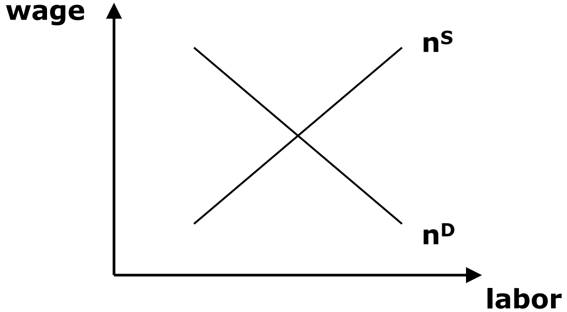

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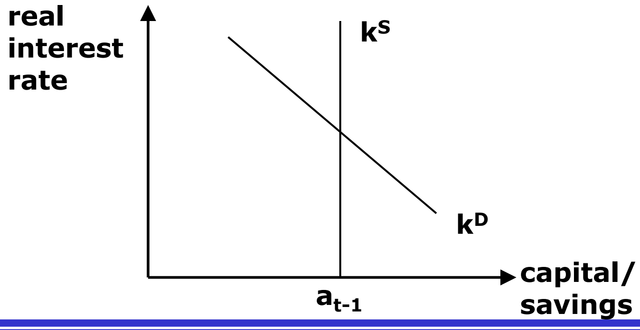

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$$k_t^D = a_{t-1} \left(= k_t^S\right)$$

- **Capital depreciates at rate *δ* each period**
  - **Economic depreciation, due to physical wear and tear of production**
  - **Not accounting depreciation**
  - **Compensation reflected in capital-market-clearing price: $r_t = r^k_t - \delta$**

CAPITAL SUPPLY

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  - **From where?**

Capital Supply

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- **Euler equation**

$$u'(c_t) = \beta E_t \left\{ u'(c_{t+1})(1 + r_{t+1}^k - \delta) \right\}$$

  - **From perspective of single individual: characterizes optimal savings (flow!) decision between *t* and *t*+1**
  - **From perspective of entire economy: characterizes optimal investment (flow!) in capital stock between *t* and *t*+1**
- **Closed economy: domestic savings = domestic investment**
- **Note timing: savings/investment decisions in *t* alter the available capital stock in period *t*+1 ("time to build")**

TOWARDS DYNAMIC GENERAL EQUILIBRIUM

- Round out final details
- Baseline model(s)
  - Consumption goods and capital goods are freely interchangeable
  - i.e., capital good in a given period can be “dismantled” and used for consumption in future periods
  - No irreversibility of investment process

Towards Dynamic General Equilibrium

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  - Corollary: factors of production are paid …?…

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  n_t defined as $n^D_t = n^S_t$, for all t (with clearing price w_t)
- Capital-market clearing

  k_t defined as $k^D_t = k^S_t$, for all t (with clearing price r^k_t)
- Goods market clearing

  $c_t + k_{t+1} - (1-\delta)k_t = z_t f(k_t,n_t)$, for all t (with clearing price = …?…)

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DYNAMIC GENERAL EQUILIBRIUM

- **Economy-wide state vector in period t: $(k_t; z_t)$**
- **Consider $T \rightarrow$ infinity**

- **Definition: a dynamic stochastic general equilibrium is time-invariant state-contingent price functions $w(k_t; z_t)$, $r^k(k_t; z_t)$ and state-contingent consumption, labor, and (one-period-ahead) capital decision rules $c(k_t; z_t)$, $n(k_t; z_t)$, and $k(k_t; z_t)$ that jointly satisfy the following:**

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  2. **(Firm optimality) Given $w(k_t; z_t)$, $r^k(k_t; z_t)$, the function $n(k_t; z_t)$ satisfies the labor demand function and k_t satisfies the capital demand function**

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  3. (Markets clear)
     - Labor-market clearing

       $n(k_t; z_t)$ defined as $n^D{}_t = n^S{}_t$, for all t
     - Capital-market clearing

       k_t defined as $k^D{}_t = k^S{}_t$, for all t
     - Goods market clearing

       $$c(k_t; z_t) + k(k_t; z_t) - (1-\delta)k_t = z_t f(k_t, n(k_t; z_t)), \text{ for all t}$$

  given the initial capital stock k_0 and (Markov) transition process for $z_t \rightarrow z_{t+1}$

INTERTEMPORAL MODELS: BASICS OF DYNAMIC PROGRAMMING

DYNAMIC PROGRAMMING

- Can we represent intertemporal problems recursively?
  - Rather than sequentially

- Benefits
  - Allows application of series of theorems/results that guarantees (conditional on model...) a solution exists in the space of functions
  - Allows application of series of theorems/results that help find solution in the space of functions
  - Computational algorithms require it – computers can't handle infinite-dimensional objects!

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  - May rule out some solutions to the original (sequential) problem
  - Requires (a lot?) more structure on the problem
  - Sometimes (often?) not obvious how to recast sequential problem as recursive problem

- Ljungqvist and Sargent (2012, Preface p. 34)

"The art in applying recursive methods is to find a convenient definition of a state. It is often not obvious what the state is, or even whether a finite-dimensional state exists."

Dynamic Programming

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- Start with deterministic case
  - (Fairly) straightforward
  - Stochastic case requires more structure

FROM SEQUENTIAL TO RECURSIVE

- **Lagrangian of consumer problem, with planning horizon *T***

$$\max_{\{c_t, a_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t \left[u(c_t) + \lambda_t \left(y_t + (1 + r_{t-1}) a_{t-1} - c_t - a_t \right) \right]$$

- **State variables of consumer problem at beginning of any period *s***
  - a_{s-1} **(accumulation variable)**
  - r_s **(price-taker)**
  - **A sufficient summary of the dynamic position of the environment in which the consumer operates**

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- **Lagrangian of consumer problem, with planning horizon *T***

$$V^0(a_{-1}, r_0;.) \equiv \max_{\{c_t, a_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t \left[u(c_t) + \lambda_t \left(y_t + (1+r_{t-1})a_{t-1} - c_t - a_t \right) \right]$$

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- **Define $V^0(a_{-1}, r_0; .)$ as value function starting from period zero**
  - **The maximized value of the constrained optimization problem**
  - **As function of period-zero parameters of the problem**

- **Goal: recast problem of finding optimal sequence $\{c_t, a_t\}_{t=0,1,2,\ldots T}$ into problem of finding functions $\{V^i(.)\}_{t=0,1,2,\ldots T}$**
  - **(Actually, find $V^i(.)$ along with two other functions)**

FROM SEQUENTIAL TO RECURSIVE

- Write out more explicitly

$$V^0(a_{-1}, r_0;.) \equiv \max_{\{c_0, a_0, c_t, a_t\}_{t=1}^{\infty}} \left\{ \begin{array}{l} u(c_0) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c_0 - a_0 \right) \\ \qquad + \sum_{t=1}^{T} \beta^t \left[u(c_t) + \lambda_t \left(y_t + (1+r_{t-1})a_{t-1} - c_t - a_t \right) \right] \end{array} \right\}$$

From Sequential to Recursive

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Separate terms

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Note the max inside the max

Adjust β factors

From Sequential to Recursive

Adjust β factors

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FROM SEQUENTIAL TO RECURSIVE

Adjust β factors $\longrightarrow$

$$V^0(a_{-1},r_0;.) \equiv \max_{c_0,a_0}\left\{u(c_0)+\lambda_0\left(y_0+(1+r_{-1})a_{-1}-c_0-a_0\right)\right\}$$

$$+\beta\cdot\max_{c_0,a_0}\left\{\max_{\{c_t,a_t\}_{t=1}^{\infty}}\sum_{t=1}^{T}\beta^{t-1}\left[u(c_t)+\lambda_t\left(y_t+(1+r_{t-1})a_{t-1}-c_t-a_t\right)\right]\right\}$$

***$V^0(a_{-1},r_0;.)$* is value function starting from period 0.**

<u>Bellman Principle of Optimality:</u> optimal decisions in the initial period induce a future state, from which (future) decisions are optimal (Bellman, 1957)

= $V^1(a_0,r_1;.)$, value function starting from period 1.

The value resulting from optimal decisions starting from period 1.

From Sequential to Recursive

$$V^0(a_{-1},r_0;.) \equiv \max_{c_0,a_0}\Big\{u(c_0)+\lambda_0\big(y_0+(1+r_{-1})a_{-1}-c_0-a_0\big)\Big\}$$

Adjust β factors $\longrightarrow$

$$+\beta\cdot\max_{c_0,a_0}\underbrace{\left\{\max_{\{c_t,a_t\}_{t=1}^{\infty}}\sum_{t=1}^{T}\beta^{t-1}\Big[u(c_t)+\lambda_t\big(y_t+(1+r_{t-1})a_{t-1}-c_t-a_t\big)\Big]\right\}}$$

$V^0(a_{-1},r_0;.)$ is value function starting from period 0.

<u>Bellman Principle of Optimality:</u> **optimal decisions in the initial period induce a future state, from which (future) decisions are optimal (Bellman, 1957)**

= $V^1(a_0,r_1;.)$, value function starting from period 1.

The value resulting from optimal decisions starting from period 1.

Recursive representation of consumer problem

$$V^0(a_{-1},r_0;.) \equiv \max_{c_0,a_0}\Big\{u(c_0)+\lambda_0\big(y_0+(1+r_{-1})a_{-1}-c_0-a_0\big)+\beta\cdot V^1(a_0,r_1;.)\Big\}$$

- **Bellman Equation**
- **Can analyze optimization problem for period zero…**
  - **…given Bellman Principle of Optimality holds**
  - **(But how do $V^0(.)$ and $V^1(.)$ relate to each other?)**

Bellman Equation

- **Bellman Equation**

$$V^0(a_{-1}, r_0; .) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 \left(y_0 + (1 + r_{-1}) a_{-1} - c_0 - a_0 \right) + \beta \cdot V^1(a_0, r_1; .) \right\}$$

  - **Starting point for recursive analysis**
  - **Applicable to finite *T*-period or *T* → ∞ problems**
  - **Construction requires identifying state variables of optimization problem**

Bellman Equation

- **Bellman Equation**

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  - **Starting point for recursive analysis**
  - **Applicable to finite *T*-period or *T* → ∞ problems**
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- ***T*-period problem**
  - **Solution involves sequence of functions $V^0(.)$, $V^1(.)$, … , $V^{T-1}(.)$, $V^T(.)$**
  - **$V^i(.)$ functions in general will differ – reflecting time until end of planning horizon**
  - **E.g., maximized value starting from age = 60 different from maximized value starting from age = 30 (intuitively)**

- **Infinite-horizon problem**
  - **Deterministic case: $V(.) = V^i(.) = V^j(.)$ all *i,j***
  - **Always an infinity of periods left to go**

Stochastic case?

Requires more structure…

BELLMAN EQUATION

- **Bellman Equation (for $T \to \infty$)**

$$V(a_{-1}, r_0; .) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 \left(y_0 + (1 + r_{-1}) a_{-1} - c_0 - a_0 \right) + \beta \cdot V(a_0, r_1; .) \right\}$$

- **Use to characterize optimal decisions**
- **Period-0 FOCs**

c_0: $u'(c_0) - \lambda_0 = 0$

a_0: $-\lambda_0 + \beta V_1(a_0, r_1; .) = 0$ **How to compute $V_1(.)$?**

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a_0: $-\lambda_0 + \beta V_1(a_0, r_1; .) = 0$ **How to compute $V_1(.)$?**

Return to this …

- ***Suppose*** **optimal choice characterized by $c_0 = c(a_{-1}; .)$, $a_0 = a(a_{-1}; .)$ ($c(.)$ and $a(.)$ time-invariant functions in infinite-period problem)**
  - **Insert in value function (can now drop max operator)**

$$V(a_{-1}, r_0; .) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1 + r_{-1}) a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1; .)$$

  - **Now compute marginal**

BELLMAN EQUATION

- **Bellman Equation (for $T \rightarrow \infty$)**

$$V(a_{-1}, r_0;.) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1;.)$$

  - **Now compute marginal (suppress *r* argument of *c*(.) and *a*(.) functions)**

$$V_1(a_{-1}, r_0;.) =$$

BELLMAN EQUATION

- **Bellman Equation (for $T \rightarrow \infty$)**

$$V(a_{-1}, r_0; .) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1 + r_{-1}) a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1; .)$$

  - **Now compute marginal (suppress *r* argument of *c*(.) and *a*(.) functions)**

$$V_1(a_{-1}, r_0; .) = u'(c_0) \cdot c'(a_{-1}) + \lambda_0 (1 + r_{-1}) - \lambda_0 \cdot c'(a_{-1}) - \lambda_0 \cdot a'(a_{-1}) + \beta V_1(a_0, r_1; .) \cdot a'(a_{-1})$$

BELLMAN EQUATION

- **Bellman Equation (for *T* → ∞)**

$$V(a_{-1},r_0;.)\equiv u(c(a_{-1}))+\lambda_0\left(y_0+(1+r_{-1})a_{-1}-c(a_{-1})-a(a_{-1})\right)+\beta\cdot V(a(a_{-1}),r_1;.)$$

  - **Now compute marginal (suppress *r* argument of *c*(.) and *a*(.) functions)**

$$V_1(a_{-1},r_0;.)=u'(c_0)\cdot c'(a_{-1})+\lambda_0(1+r_{-1})-\lambda_0\cdot c'(a_{-1})-\lambda_0\cdot a'(a_{-1})+\beta V_1(a_0,r_1;.)\cdot a'(a_{-1})$$

$$=\lambda_0(1+r_{-1})+\underbrace{\left[u'(c_0)-\lambda_0\right]}_{= 0 \text{ by period-0 FOCs}}\cdot c'(a_{-1})+\underbrace{\left[-\lambda_0+\beta V_1(a_0,r_1;.)\right]}_{= 0 \text{ by period-0 FOCs}}\cdot a'(a_{-1})$$

BELLMAN EQUATION

- **Bellman Equation (for $T \rightarrow \infty$)**

$$V(a_{-1}, r_0;.) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1;.)$$

  - **Now compute marginal (suppress *r* argument of *c*(.) and *a*(.) functions)**

$$\begin{aligned} V_1(a_{-1}, r_0;.) &= u'(c_0) \cdot c'(a_{-1}) + \lambda_0(1+r_{-1}) - \lambda_0 \cdot c'(a_{-1}) - \lambda_0 \cdot a'(a_{-1}) + \beta V_1(a_0, r_1;.) \cdot a'(a_{-1}) \\ &= \lambda_0(1+r_{-1}) + \underbrace{\left[u'(c_0) - \lambda_0\right]}_{\text{= 0 by period-0 FOCs}} \cdot c'(a_{-1}) + \underbrace{\left[-\lambda_0 + \beta V_1(a_0, r_1;.)\right]}_{\text{= 0 by period-0 FOCs}} \cdot a'(a_{-1}) \end{aligned}$$

$$\Rightarrow \quad V_1(a_{-1}, r_0;.) = \lambda_0(1+r_{-1})$$

Envelope Condition

- **Envelope Theorem**

Note: envelope theorem has nothing to do with dynamic programming

  - **In computing first-order effects of changes in a problem's parameters on the maximized value, can ignore how optimal choices will adjust**
    - **Intuition: because already at a max (marginal costs = marginal benefits)**
  - **Need only consider the direct effect**

Bellman Equation

- **Bellman Equation (for $T \to \infty$)**

$$V(a_{-1}, r_0; .) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1; .)$$

- **Now compute marginal (suppress *r* argument of *c*(.) and *a*(.) functions)**

$$V_1(a_{-1}, r_0; .) = u'(c_0) \cdot c'(a_{-1}) + \lambda_0 (1+r_{-1}) - \lambda_0 \cdot c'(a_{-1}) - \lambda_0 \cdot a'(a_{-1}) + \beta V_1(a_0, r_1; .) \cdot a'(a_{-1})$$

$$= \lambda_0 (1+r_{-1}) + \underbrace{\left[u'(c_0) - \lambda_0 \right]}_{= 0 \text{ by period-0 FOCs}} \cdot c'(a_{-1}) + \underbrace{\left[-\lambda_0 + \beta V_1(a_0, r_1; .) \right]}_{= 0 \text{ by period-0 FOCs}} \cdot a'(a_{-1})$$

$$\Rightarrow V_1(a_{-1}, r_0; .) = \lambda_0 (1+r_{-1}) \xrightarrow{\text{evaluate at period 1}} V_1(a_0, r_1; .) = \lambda_1 (1+r_0)$$

Envelope Condition

- **Envelope Theorem**

Note: envelope theorem has nothing to do with dynamic programming

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  - **Intuition: because already at a max (marginal costs = marginal benefits)**
- **Need only consider the direct effect**

Bellman Equation

- **Bellman Equation (for *T* → ∞)**

$$V(a_{-1}, r_0; .) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1; .)$$

- **Use to characterize optimal decisions**

- **Period-0 FOCs, now evaluated using *c*(*a*-1), *a*(*a*-1)**

c_0: $u'(c(a_{-1})) - \lambda_0 = 0$

a_0: $-\lambda_0 + \beta V_1(a_0(a_{-1}), r_1; .) = 0$

Env: $V_1(a(a_{-1}), r_1; .) = \lambda_1 (1 + r_0)$

Bellman Equation

- **Bellman Equation (for *T* → ∞)**

$$V(a_{-1},r_0;.)\equiv u(c(a_{-1}))+\lambda_0\left(y_0+(1+r_{-1})a_{-1}-c(a_{-1})-a(a_{-1})\right)+\beta\cdot V(a(a_{-1}),r_1;.)$$

- **Use to characterize optimal decisions**

- **Period-0 FOCs, now evaluated using *c(a$_{-1}$), a(a$_{-1}$)***

$$\left.\begin{array}{ll}\textbf{c}_0\textbf{:} & u'(c(a_{-1}))-\lambda_0=0\\ \textbf{a}_0\textbf{:} & -\lambda_0+\beta V_1(a_0(a_{-1}),r_1;.)=0\\ \textbf{Env:} & V_1(a(a_{-1}),r_1;.)=\lambda_1(1+r_0)\end{array}\right\}\quad u'(c(a_{-1}))=\beta(1+r_0)u'(c(a_0))$$

- **Seems like usual Euler equation from sequential analysis (deterministic)…**

DETERMINISTIC – RECURSIVE ANALYSIS

- **Solution of infinite-horizon consumer problem (starting from date zero)…**
- **…is a consumption decision rule *c*(a_{-1};.), asset decision rule *a*(a_{-1};.), and value function *V*(a_{-1};.) that satisfies**
  - **Bellman equation**

$$V(a_{-1},r_0;.) \equiv u(c(a_{-1})) + \lambda_0\left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1})\right) + \beta \cdot V(a(a_{-1}), r_1;.)$$

  - **Euler equation**

$$u'(c(a_{-1})) = \beta V_1(a(a_{-1}), r_1;.) \quad \xleftrightarrow{\text{by envelope theorem}} \quad u'(c(a_{-1})) = \beta(1+r_0)u'(c(a_0))$$

    - **which is the TVC in the limit *t* → ∞:** $\lim\limits_{t\to\infty} \beta^t u'(c(a^*_{t-1}))\cdot a(a^*_{t-1}) = 0$

  - **Budget constraint**

$$y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) = 0$$

taking as given $\left(a_{-1}, r_0, r_{-1}\right)$

Deterministic – Sequential Analysis

- **Solution of infinite-horizon consumer problem (starting from date zero)…**
- **is a consumption and asset sequence** $\left\{c_t^*, a_t^*\right\}_{t=0}^{\infty}$ **that satisfies**
  - **Sequence of flow budget constraints**

$$c_t^* + a_t^* = y_t + (1 + r_{t-1})a_{t-1}^*, \quad t = 0, 1, 2, \ldots$$

  - **Sequence of Euler equations**

$$u'(c_t^*) = \beta u'(c_{t+1}^*)(1 + r_t), \quad t = 0, 1, 2, \ldots$$

    - **which is the TVC in the limit** $t \to \infty$**:** $\lim_{t \to \infty} \beta^t u'(c_t^*) a_t^* = 0$

taking as given $\left(\left\{r_t, y_t\right\}_{t=0}^{\infty}, a_{-1}, r_{-1}\right)$

Does solution to recursive problem coincide with solution to sequential problem?

Recursive vs. Sequential Analysis

- **Does solution to recursive problem coincide with solution to sequential problem?**
- **Does solution to sequential problem coincide with solution to recursive problem?**

- **In general, no!**
- **No reason why it should!**

RECURSIVE VS. SEQUENTIAL ANALYSIS

- Does solution to recursive problem coincide with solution to sequential problem?
- Does solution to sequential problem coincide with solution to recursive problem?

- In general, no!
- No reason why it should!

- In constructing Bellman representation ($T \rightarrow \infty$ case), the imposition of time-invariant functions $c(a)$, $a(a)$ potentially limits the class of solutions
  - In original sequential formulation, this is neither explicitly nor implicitly a requirement of the solution!

- In general (here without proof…)

  - Solution to the sequential problem is also a solution to the recursive problem

  - Solution to the recursive problem is also a solution to the sequential problem provided some further regularity conditions hold

- Stokey, Lucas, Prescott text (1989)

RECURSIVE VS. SEQUENTIAL ANALYSIS

- So why go recursive?
  - Allows application of series of theorems/results that guarantee a **solution exists in the space of functions**
  - Allows application of series of theorems/results that help **find solution in the space of functions**

Underlying theory:
Contraction Mapping Theorem, Blackwell's Sufficient Conditions for a Contraction, Theorem of the Maximum

THEORY

- **Blackwell's Sufficient Conditions for a Contraction:** **Let X be a subset of R^l and let $B(X)$ be the set of bounded functions $f : X \rightarrow R$ with the sup norm. Let $T : B(X) \rightarrow B(X)$ be an operator satisfying**

  a. **(Monotonicity)** $f, g \in B(X)$ **and** $f(x) \leq g(x)$**, for all** $x \in X$**, implies** $(Tf)(x) \leq (Tg)(x)$**, for all** $x \in X$

  b. **(Discounting) There exists some** $\beta \in (0,1)$ **such that**

  $[T(f+a)](x) \leq (Tf)(x) + \beta a$**, for all** $f \in B(X)$, $a \geq 0$, $x \in X$

  Then T is a contraction with modulus β.

  (Note: $(f+a)(x)$ **is the function defined by** $(f+a)(x) = f(x) + a$**)**

THEORY

- Let (S, ρ) be a metric space and $T : S \rightarrow S$ be a function mapping set S into itself. T is a **contraction mapping (with modulus β)** if for some $\beta \in (0,1)$, $\rho(Tx, Ty) \leq \beta\rho(x, y)$ for all $x, y \in S$.
  Example: $S = [a, b]$ with $\rho(x, y) = |x - y|$ (Euclidean norm)

- **Contraction Mapping Theorem:** If (S, ρ) is a metric space and $T : S \rightarrow S$ is a contraction mapping with modulus β, then
  a. T has exactly one fixed point v in set S.
  b. For any $v_0 \in S$, $\rho(T^n v_0, v) \leq \beta\rho(x, y)$ for $n = 0, 1, 2, \ldots$

- CMT states that a contraction mapping has a **unique fixed point,** and the fixed point can be found by iterative application of the mapping T starting starting from any point in S.

THEORY

- **General class of problems to which our (usual) economic optimization problems belong have the form**

  $$(Tv)(x) = \sup_{y \in \Gamma(x)} [F(x,y) + \beta v(y)]$$

- **For our economic theory: would like operator *T* to map the space *C*(*X*) of bounded continuous functions of the state vector into itself. Would also like to be able to characterize the set of maximizing values of *y* given *x*.**

THEORY

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- For our economic theory: would like operator T to map the space $C(X)$ of bounded continuous functions of the state vector into itself. Would also like to be able to characterize the set of maximizing values of y given x.

- **Theorem of the Maximum:** Let X be a subset of R^l, Y be a subset of R^m, let $f : X \times Y \rightarrow R$ be a (single-valued) continuous function, and let $\Gamma : X \rightarrow Y$ be a compact-valued and continuous correspondence. The problem we are interested in is of the form $\sup_{y \in \Gamma(x)} f(x,y)$. Then

  a. *sup* can be replaced with *max* because, for each *x,* the maximum is attained and the function $h(x) = \max_{y \in \Gamma(x)} f(x,y)$ is well defined and continuous

  b. The correspondence $G(x) = y \in \Gamma(x) : f(x,y) = h(x)$ is well defined, is non-empty, is compact-valued, and upper hemi-continuous.

- Theorem of the Maximum establishes the **existence** of the maximum of the problem.

THEORY

- **Suppose in addition to the hypotheses of the Theorem of the Maximum, the correspondence *Γ* is convex-valued and the function *f* is strictly concave in *y*.**

  → Then *G* is single-valued. Call this function *g*, and *g* is continuous.

- **Establishes that, given these conditions and given the unique solution of the Bellman Equation, there is a unique *g* that is the optimal "decision rule."**

- **If $\{f_n(x,y)\}$ is a sequence of continuous functions converging to $f(x,y)$, each strictly concave in *y*, then the sequence of functions $\{g_n(x)\}$ (which are the argmax of the sequence $\{f_n(x,y)\}$) converges pointwise to $g(x)$, which is the argmax of $f(x,y)$.**

- **The latter result is very useful considered in the context of the Contraction Mapping Theorem. It guarantees that the solutions to the sequence of problems converges to the true solution.**

RECURSIVE VS. SEQUENTIAL ANALYSIS

- So why go recursive?
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  - Allows application of series of theorems/results that help find solution in the space of functions

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  - Computational algorithms require it – computers can't handle infinite-dimensional objects!

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- Can't really "choose" whether want to analyze problem sequentially or recursively
  - All but the most limited of problems/models require computational solution
  - In which case model analysis is recursive

- What about stochastic dynamic programming?
  - Even more structure required....

STOCHASTIC DYNAMIC PROGRAMMING

- **Even more structure required on the problem to recursively solve dynamic stochastic optimization problems**

- **Main (new) technical problem**
  - **Branching of event tree at each of *T* periods (possibly $T \rightarrow \infty$)**

- **Main technical solution/assumption**
  - **Assume risk follows Markov process**
  - **Which enables series of theorems/results from deterministic dynamic programming to work in stochastic case…**
  - **…given further technical regularity assumptions**

Recursive Representation

- **State variables**
  - **A sufficient summary, as of the start of period *t*, of the dynamic position of the environment in which the maximizing agent operates**
  - **"Environment" of the agent – what needs to be known in order to optimize in period *t*?**
    - **Individual-specific quantities**
    - **Market prices**
    - **Government policies**
    - **(Fixed structural parameters – will omit from state vector for parsimony)**

The usual suspects

Important: states can be endogenous or exogenous

- **Ljungqvist and Sargent (2012, Preface p. 34)**

"The art in applying recursive methods is to find a convenient definition of a state. It is often not obvious what the state is, or even whether a finite-dimensional state exists."

RECURSIVE REPRESENTATION

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The usual suspects

    - **Individual-specific quantities**
    - **Market prices**
    - **Government policies**

Important: states can be endogenous or exogenous

    - **(Fixed structural parameters – will omit from state vector for parsimony)**
  - **"Sufficient" – there are no other objects (quantities, prices, govt policies, etc.) that must be known in order to optimize in period *t***
  - **Concept well-defined for both finite-*T* and *T* → ∞ problems**
  - **KEY: Period-*t* decisions are function of the period-*t* state variables**
- **Ljungqvist and Sargent (2012, Preface p. 34)**

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- **Bellman Equation**

$$V^0(a_{-1}, r_0; .) \equiv \max_{c_0, a_0} \left\{ u(c_0) + \lambda_0 \left(y_0 + (1 + r_{-1}) a_{-1} - c_0 - a_0 \right) + \beta \cdot V^1(a_0, r_1; .) \right\}$$

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  - **Solution involves sequence of functions $V^0(.)$, $V^1(.)$, … , $V^{T-1}(.)$, $V^T(.)$**
  - **$V^i(.)$ functions in general will differ – reflecting time until end of planning horizon**
  - **E.g., maximized value starting from age = 60 different from maximized value starting from age = 30 (intuitively)**

- **Infinite-horizon problem (“stationary” environment)**
  - **Deterministic case: $V(.) = V^i(.) = V^j(.)$ all *i,j***
  - **Always an infinity of periods left to go**

BELLMAN EQUATION

- **Bellman Equation (for *T* → ∞)**

$$V(a_{-1},r_0;.) \equiv \max_{c_0,a_0}\left\{u(c_0)+\lambda_0\left(y_0+(1+r_{-1})a_{-1}-c_0-a_0\right)+\beta\cdot V(a_0,r_1;.)\right\}$$

- **Use to characterize optimal decisions**

- **Period-0 FOCs, evaluated using time-invariant *c(a₋₁), a(a₋₁)***

$$\left.\begin{array}{ll} \boldsymbol{c_0}\text{:} & u'(c(a_{-1}))-\lambda_0=0 \\ \boldsymbol{a_0}\text{:} & -\lambda_0+\beta V_1(a_0(a_{-1}),r_1;.)=0 \\ \textbf{Env:} & V_1(a(a_{-1}),r_1;.)=\lambda_1(1+r_0) \end{array}\right\} \quad u'(c(a_{-1}))=\beta(1+r_0)u'(c(a_0))$$

- **Seems like usual Euler equation from sequential analysis (deterministic)…**

BELLMAN EQUATION

- **Bellman Equation (for $T \rightarrow \infty$)**

$$V(a_{-1}, r_0; .) \equiv u(c(a_{-1})) + \lambda_0 \left(y_0 + (1+r_{-1})a_{-1} - c(a_{-1}) - a(a_{-1}) \right) + \beta \cdot V(a(a_{-1}), r_1; .)$$

- **Seems like a two-period problem**
  - **In terms of (value) functions, not in terms of choice variables**
  - **Optimize in current period**
  - **Optimize next period (Bellman's Principle of Optimality)**

Bellman Equation

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$$V(a_{-1},r_0;.)\equiv u(c(a_{-1}))+\lambda_0\left(y_0+(1+r_{-1})a_{-1}-c(a_{-1})-a(a_{-1})\right)+\beta\cdot V(a(a_{-1}),r_1;.)$$

- **Seems like a two-period problem**
  - **In terms of (value) functions, not in terms of choice variables**
  - **Optimize in current period**
  - **Optimize next period (Bellman's Principle of Optimality)**
- **Common notation**
  - **Use *x* for current-period variables**
  - **Use *x'* for next-period variables**
- **Bellman Equation**

$$V(a,r;.)\equiv u(\underbrace{c(a)}_{=c})+\lambda\left(y+(1+r_{-1})a-\underbrace{c(a)}_{=c}-\underbrace{a(a)}_{=a'}\right)+\beta\cdot V(\underbrace{a(a)}_{=a'},r';.)$$

- **Euler equation**

$$u'(\overbrace{c(a)}^{=c})=\beta(1+r)u'(\overbrace{c(a')}^{=c'})$$

RECURSIVE VS. SEQUENTIAL ANALYSIS

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  - Allows application of series of theorems/results that help find solution in the space of functions

Underlying Theory:
Contraction Mapping Theorem, Blackwell's Sufficient Conditions for a Contraction, Theorem of the Maximum

- Suppose *V*(.) exists
- VFI: Procedure for finding *V*(.) and associated decision rules: iterate on Bellman Equation starting from any arbitrary initial guess

Recursive vs. Sequential Analysis

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initial guess (some parametric form)

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If no, insert $V^2(.)$ on RHS and repeat

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  - **All but the most limited of problems require computational solution**
  - **In which case model analysis is recursive**

- **"Solving model sequentially"**
  - **Doesn't seem recursive…** $\longrightarrow$ $u(c_t) = \beta(1+r_t)u'(c_{t+1})$

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- **"Solving model sequentially"**
  - **Doesn't seem recursive…**
  - **…but computational implementation requires time-invariant decision rule**

$$u(c_t) = \beta(1+r_t)u'(c_{t+1})$$

Imposing recursivity on solution

$$u(c(a_{t-1})) = \beta(1+r_t)u'(c(a_t))$$

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- What about stochastic dynamic programming?
  - Even more structure required….
  - The key assumption is Markov risk

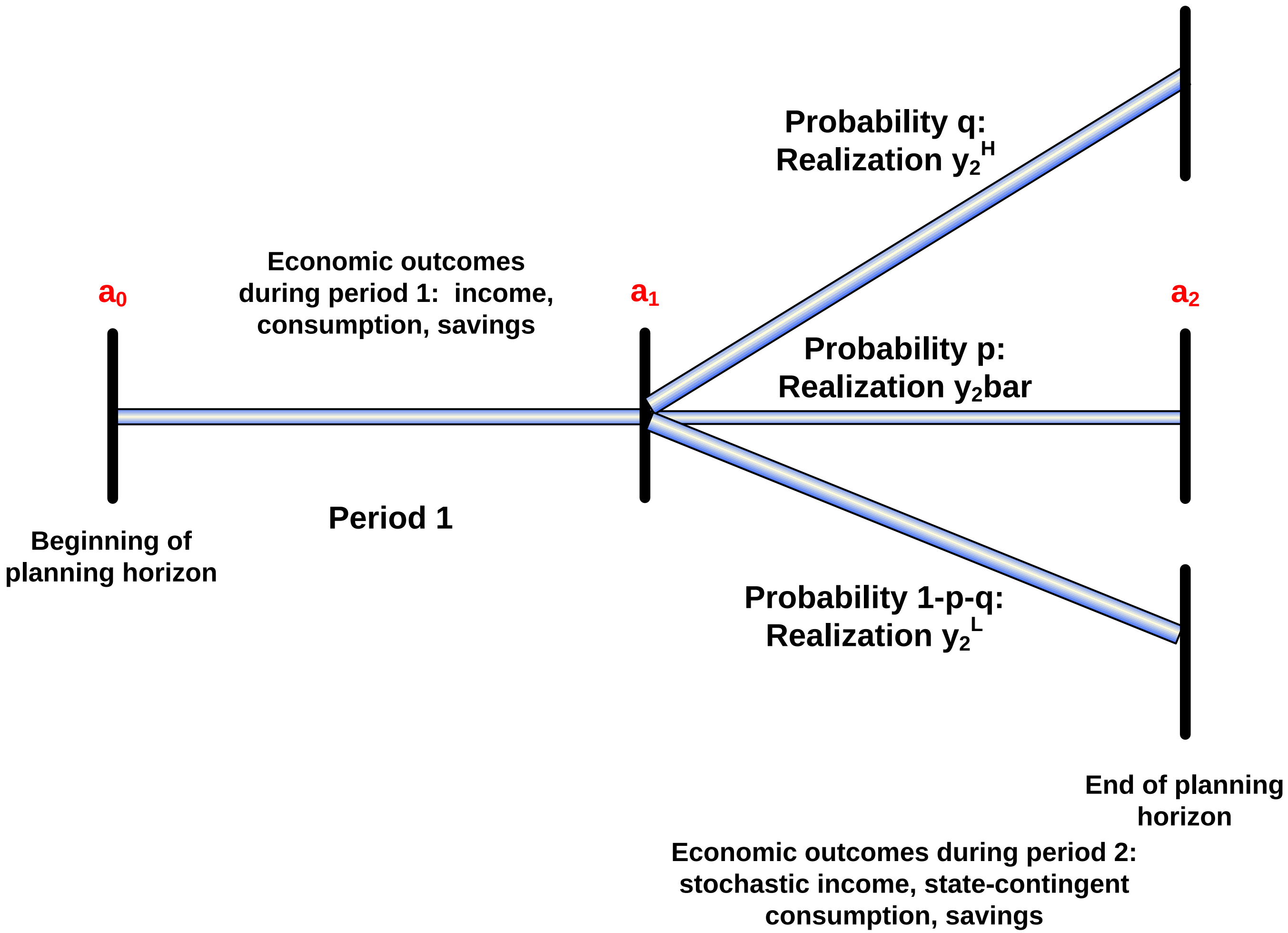
STOCHASTIC DYNAMIC PROGRAMMING

- **Even more structure required on the problem to recursively solve dynamic stochastic optimization problems**

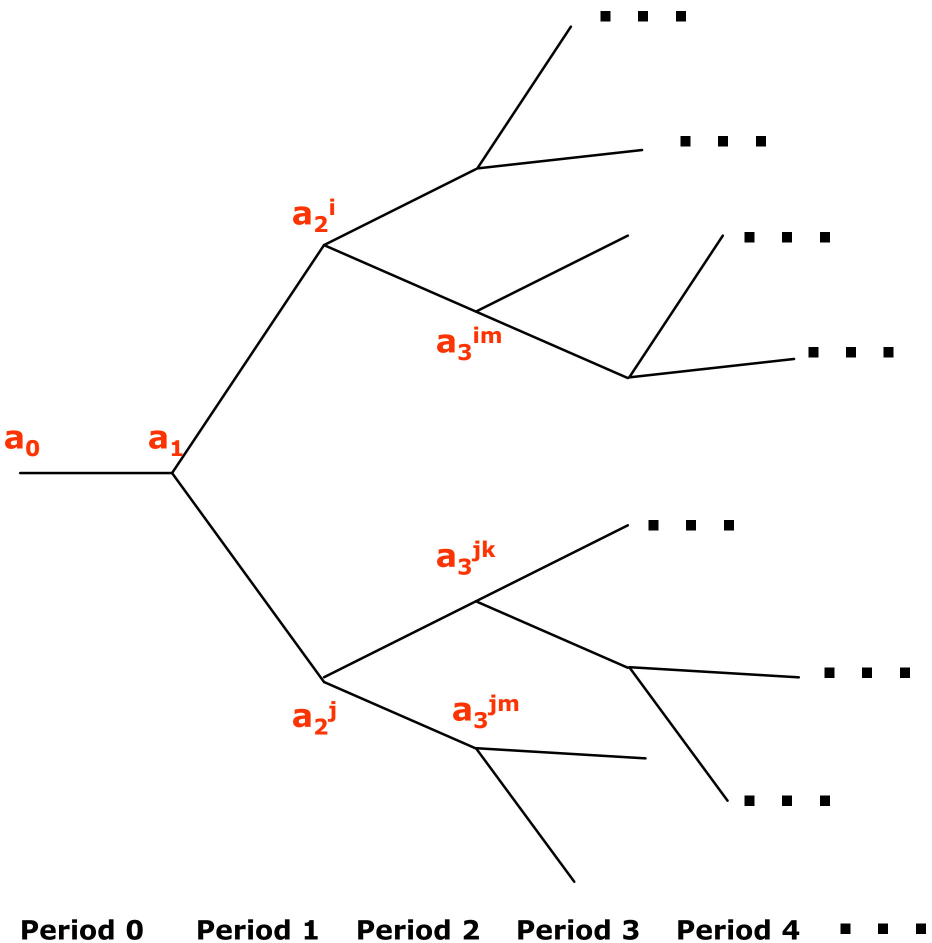
- **Main (new) technical problem**
  - **Branching of event tree at each of *T* periods (possibly *T* → ∞)**

- **Main technical solution/assumption**
  - **Assume risk follows Markov process**
  - **Which enables series of theorems/results from deterministic dynamic programming to work in stochastic case…**
  - **…given further technical regularity assumptions**

EVENT TREE


Probability q:
Realization y_2^H
a_0
Economic outcomes during period 1: income, consumption, savings
a_1
a_2
Probability p:
Realization $y_2$bar
Period 1
Beginning of planning horizon
Probability 1-p-q:
Realization y_2^L
End of planning horizon
Economic outcomes during period 2: stochastic income, state-contingent consumption, savings

EVENT TREE



- In general could be any arbitrary unfolding of exogenous risk

  e.g., if state *i* realized in period 16, then 14 possible states in period 17; but if state *j* realized in period 16, then 8 possible states in period 17

  OR

  e.g., probability of state *i* in period *t* depends on event in period *t*-100000

- Number of decision rules to solve explodes
- Intractable!!!
- "Curse of dimensionality"
- Requires a lot of structure on exogenous risk

RISK STRUCTURE

Assumptions

- Set of realizations of exogenous state variable is independent of date

$$S_2 = S_3 = S_4 = S_5 = \ldots. = S_{T-1} = S_T$$

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- Set of realizations of exogenous state variable is independent of date
- Probability of realization of exogenous state variable in period *t* depends only on outcomes in period *t*-1
  - Suppose X_t is a stochastic process and x_t is a particular realization
  - X_t is a Markov process if

$$\Pr\left(X_t = x_t \mid X_{t-1} = x_{t-1}, X_{t-2} = x_{t-2}, X_{t-3} = x_{t-3}, \ldots., X_{t-10000} = x_{t-10000}, \ldots.\right)$$

$$= \Pr\left(X_t = x_t \mid X_{t-1} = x_{t-1}\right)$$ CONDITIONAL probability depends on only *t*-1

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  - Not as restrictive as it may seem – could have finite lags in process
  - E.g.

  - Just can't have infinite lags (in principle) or "too many" (finite) lags (in computational practice)

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- Exogenous state variable is Markov process + assumption/result that decision rules are time-invariant (for $T \rightarrow \infty$) functions of state variables

$\Rightarrow$ **Endogenous processes are Markov**

given several regularity assumptions

Underlying theory:

Stokey, Lucas, Prescott (1989, Chapters 8-12)

STOCHASTIC – SEQUENTIAL ANALYSIS

- **Planning horizon *T* → ∞**
- **Exogenous state drawn from set *S* (could be continuous or discrete)**
- **Suppose single asset with state-contingent *r* (will illustrate main ideas)**

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- **FOCs**

***c_0*:**

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***c_1*:**

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***a*₀:** $-\lambda_0 + \beta E_0\left[\lambda_1(1+r_1)\right] = 0 \longrightarrow 1 = E_0\left[\dfrac{\beta\lambda_1}{\lambda_0}(1+r_1)\right]$

Just as in stochastic two-period model

***c*₁:** $\beta E_0 u'(c_1) - \beta E_0 \lambda_1 = 0 \longrightarrow E_0 u'(c_1^j) = E_0 \lambda_1^j, \ \forall j \in S$

Holds for each state

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  - **$E_t X_{t+2} = E_t [E_{t+1} X_{t+2}]$**

$$E_0\lambda_1 = \beta E_0\left[\lambda_2(1+r_2)\right] \longrightarrow E_0\lambda_1 = \beta E_0\left[E_1\left(\lambda_2(1+r_2)\right)\right]$$

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$$\longrightarrow E_1\lambda_1 = \beta E_1\left[\lambda_2(1+r_2)\right] \longrightarrow \lambda_1 = \beta E_1\left[\lambda_2(1+r_2)\right]$$

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c_2: $\beta^2 E_0 u'(c_2) - \beta^2 E_0 \lambda_2 = 0$ → **Holds for each state** → $E_2 u'(c_2^j) = E_2 \lambda_2^j, \ \forall j \in S$

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c_{t+1}: $\beta^{t+1} E_0 u'(c_{t+1}) - \beta^{t+1} E_0 \lambda_{t+1} = 0$ → **Holds for each date and state** → $u'(c_2^j) = \lambda_2^j, \ \forall j \in S$

- **One-period-ahead conditional expectation governs stochastic Euler condition**

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- **Denote exogenous state variables as z (e.g., $z_t = [y_t, r_t]$)**

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  - (Stochastic) Euler equation

$$u'(c(a,z)) = \beta E\left[u'(c(a',z'))(1+r')\right]$$

    - which is the (expectational) TVC in the limit $t \to \infty$:

$$\lim_{t\to\infty} E_0 \beta^t u'(c(a,z)) \cdot a(a,z) = 0$$

  - Budget constraint

$$y + (1+r)a - c(a,z) - a(a,z) = 0$$

  - Bellman Equation

Expectation in Bellman Equation

Transition from $z \to z'$

$$V(a,z;.) \equiv u(c(a,z)) + \lambda\big(y + (1+r)a - c(a,z) - a(a,z)\big) + \beta \cdot \underset{\text{Expectation in Bellman Equation}}{EV}\big(a(a,z), \underset{\text{Transition from } z \to z'}{z(a,z)};.\big)$$

taking as given (y, a, r) **and (Markov) transition function for** $z \to z'$

BELLMAN EQUATION

- **Bellman Equation (for *T* → ∞)**

$$V(a,z;.) \equiv \max_{c,a'} \left\{ u(c) + \lambda \left(y + (1+r)a - c - a' \right) + \beta \cdot EV(a',z';.) \right\}$$

Expectation in Bellman Equation

Transition from *z* → *z*'

- **Use to characterize optimal decisions**

- **Current-period FOCs, evaluated using *c(a,z;.)*, *a(a,z;.)***

***c*:**

***a*':**

Env:

Bellman Equation

- Bellman Equation (for $T \to \infty$)

$$V(a,z;.) \equiv \max_{c,a'} \left\{ u(c) + \lambda\left(y+(1+r)a-c-a'\right) + \beta \cdot EV(a',z';.) \right\}$$

Expectation in Bellman Equation

Transition from $z \to z'$

- Use to characterize optimal decisions

- Current-period FOCs, evaluated using *c(a,z;.), a(a,z;.)*

c: $u'(c(a,z)) - \lambda = 0$

a': $-\lambda + \beta EV_1(a(a,z), z(a,z);.) = 0$

Env: $EV_1(a,z;.) = \lambda(1+r)$

$$u'(c(a,z)) = \beta E\left[u'(c(a,z))(1+r)\right]$$

- Bellman analysis goes through as in deterministic case
  - (Given further technical conditions – see SLP)

MARKOV RISK

- Why does Markov assumption make everything work?
- Main issue in moving from deterministic dynamic programming to stochastic dynamic programming: preserving recursivity
  - So exogenous states must also have recursive structure
- Shocks that have this recursive structure are Markov processes
- Markov has property that given the current realization, future realizations are independent of the past
  - "Limited history dependence"
  - "Finite memory"

MARKOV RISK

- **Why does Markov assumption make everything work?**
- **Main issue in moving from deterministic dynamic programming to stochastic dynamic programming: preserving recursivity**
  - **So exogenous states must also have recursive structure**
- **Shocks that have this recursive structure are Markov processes**
- **Markov has property that given the current realization, future realizations are independent of the past**
  - **"Limited history dependence"**
  - **"Finite memory"**
- **In environments in which the "regularity conditions" that ensure standard Bellman analysis applies to stochastic problems are not satisfied…**
- **…often simply need to ASSUME decision rules are Markov to make progress**