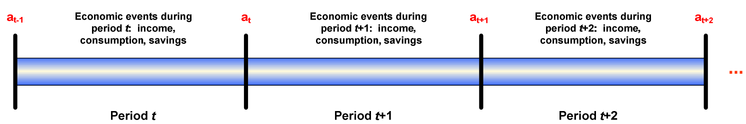
INFINITE-PERIOD CONSUMER ANALYSIS (CONTINUED)

FEBRUARY 27, 2012

BASICS

- **Timeline of events**



- **Notation**
  - c_t: consumption in period t
  - P_t: nominal price of consumption in period t
  - Y_t: nominal income in period t ("falls from the sky")
  - a_{t-1}: real wealth (stock) holdings at beginning of period t / end of period t-1
  - S_t: nominal price of a unit of stock in period t
  - D_t: nominal dividend paid in period t by each unit of stock held at the start of t

  The "defining features" of stock

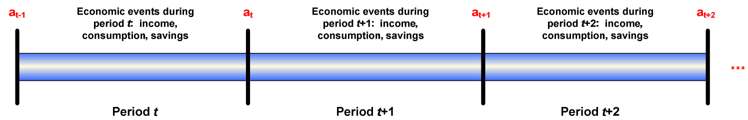
  - π_{t+1}: net inflation rate between period t and period t+1

$$\pi_{t+1} = \frac{P_{t+1} - P_t}{P_t} \left(= \frac{P_{t+1}}{P_t} - 1 \right)$$

  - y_t: real income in period t (= Y_t / P_t)

BASICS

- **Timeline of events**



- **Notation**
  - c_{t+1}: consumption in period $t+1$
  - P_{t+1}: nominal price of consumption in period $t+1$
  - Y_{t+1}: nominal income in period $t+1$ ("falls from the sky")
  - a_t: real wealth (stock) holdings at beginning of period $t+1$ / end of period t
  - S_{t+1}: nominal price of a unit of stock in period $t+1$
  - D_{t+1}: nominal dividend paid in period t by each unit of stock held at the <u>start</u> of $t+1$
  - π_{t+2}: net inflation rate between period $t+1$ and period $t+2$

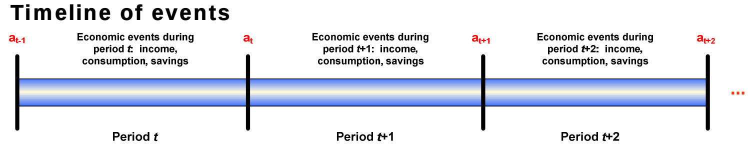
$$\pi_{t+2} = \frac{P_{t+2} - P_{t+1}}{P_{t+1}} \left(= \frac{P_{t+2}}{P_{t+1}} - 1 \right)$$

  - y_{t+1}: real income in period $t+1$ ($= Y_{t+1} / P_{t+1}$)

The "defining features" of stock

BASICS

- **Timeline of events**



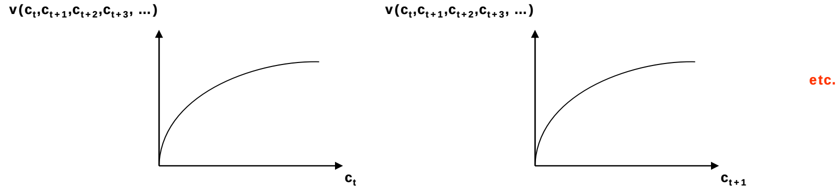
- **Notation**
  - And so on for period $t+2$, $t+3$, etc…

SUBJECTIVE DISCOUNT FACTOR

- **Infinite number of periods a more serious view of time**
- **Impatience potentially an issue when taking a serious view of time**
- **Individuals (i.e., consumers) are impatient**
  - All else equal, would rather have *X* utils today than identical *X* utils at some future date
  - An introspective statement about the world
  - An empirical statement about the world
- **Subjective discount factor**
  - A simple model of consumer impatience
  - β (a number between zero and one) measures impatience
    - The lower is β, the less does individual value future utility
  - Simple assumption about how "impatience" builds up over time
    - Multiplicatively: i.e., discount one period ahead by β, discount two periods ahead by β^2, discount three periods ahead by β^3, etc.
    - Do individuals' impatience really build up over time in this way?...limited empirical evidence so really don't know…

UTILITY

- **Preferences $v(c_t, c_{t+1}, c_{t+2}, \ldots)$ with all the "usual properties"**
  - Lifetime utility function
  - Strictly increasing in $c_t, c_{t+1}, c_{t+2}, c_{t+3}, \ldots$
  - Diminishing marginal utility in $c_t, c_{t+1}, c_{t+2}, c_{t+3}, \ldots$



- **Lifetime utility function additively-separable across time (a simplifying assumption), starting at time *t***

$$v(c_t, c_{t+1}, c_{t+2}, c_{t+3}, \ldots) = u(c_t) + \beta u(c_{t+1}) + \beta^2 u(c_{t+2}) + \beta^3 u(c_{t+3}) + \ldots$$

- **Utility side of infinite-period framework no different than Chapter 1 model – except not as simple to represent graphically**

Budget Constraint(s)

- **Suppose again *Y* "falls from the sky"**
  - **Y_t in period *t*, Y_{t+1} in period *t*+1, Y_{t+2} in period *t*+2, etc.**
- **Need infinite budget constraints to describe economic opportunities and possibilities**
  - **One for each period**
  - **Period-*t* budget constraint**

$$\underbrace{P_t c_t + S_t a_t}_{} = \underbrace{Y_t + S_t a_{t-1} + D_t a_{t-1}}_{}$$

Total expenditure in period *t*: period-*t* consumption + wealth to *carry into period t*+1

Total income in period *t*: period-*t* *Y* + income from stock-holdings *carried into period t* (has value S_t and pays dividend D_t)

  - **Period *t*+1 budget constraint**

$$\underbrace{P_{t+1} c_{t+1} + S_{t+1} a_{t+1}}_{} = \underbrace{Y_{t+1} + S_{t+1} a_t + D_{t+1} a_t}_{}$$

Total expenditure in period *t*+1: period-*t*+1 consumption + wealth to *carry into period t*+2

Total income in period *t*+1: period-*t*+1 *Y* + income from stock-holdings *carried into period t*+1 (has value S_{t+1} and pays dividend D_{t+1})

Budget Constraint(s)

- **Suppose again *Y* "falls from the sky"**
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$$\underbrace{P_t c_t + S_t a_t}_{} = \underbrace{Y_t + S_t a_{t-1} + D_t a_{t-1}}_{} \quad \longleftrightarrow \text{ can rewrite as } \quad P_t c_t + \overbrace{S_t (a_t - a_{t-1})}^{} = Y_t + D_t a_{t-1}$$

Savings during period *t* (a flow)

Dividend income during period *t* (a flow)

Total expenditure in period *t*: period-*t* consumption + wealth to *carry into period t*+1

Total income in period *t*: period-*t* *Y* + income from stock-holdings *carried into period t* (has value S_t and pays dividend D_t)

  - **Period *t*+1 budget constraint**

$$\underbrace{P_{t+1} c_{t+1} + S_{t+1} a_{t+1}}_{} = \underbrace{Y_{t+1} + S_{t+1} a_t + D_{t+1} a_t}_{} \quad \longleftrightarrow \text{ can rewrite as } \quad P_{t+1} c_{t+1} + \overbrace{S_{t+1} (a_{t+1} - a_t)}^{} = Y_{t+1} + D_{t+1} a_t$$

Savings during period *t*+1 (a flow)

Dividend income during period *t*+1 (a flow)

Total expenditure in period *t*+1: period-*t*+1 consumption + wealth to *carry into period t*+2

Total income in period *t*+1: period-*t*+1 *Y* + income from stock-holdings *carried into period t*+1 (has value S_{t+1} and pays dividend D_{t+1})

And identical-looking budget constraints for *t*+2, *t*+3, *t*+4, etc…

Lagrange Analysis: Sequential Approach

- **Step 1: Construct Lagrange function (starting from *t*)**

IMPORTANT: Discount factor β multiplies ***both*** future utility ***and*** future budget constraints

Everything (utility ***and*** income) about the future is discounted

$u(c_t)+\beta u(c_{t+1})+\beta^2 u(c_{t+2})+\beta^3 u(c_{t+3})+\ldots$	First the lifetime utility function….
$+\lambda_t\left[Y_t+(S_t+D_t)a_{t-1}-P_t c_t-S_t a_t\right]$	…then the period *t* constraint…
$+\beta\lambda_{t+1}\left[Y_{t+1}+(S_{t+1}+D_{t+1})a_t-P_{t+1}c_{t+1}-S_{t+1}a_{t+1}\right]$	…then the period *t*+1 constraint…
$+\beta^2\lambda_{t+2}\left[Y_{t+2}+(S_{t+2}+D_{t+2})a_{t+1}-P_{t+2}c_{t+2}-S_{t+2}a_{t+2}\right]$	…then the period *t*+2 constraint…
$+\beta^3\lambda_{t+3}\left[Y_{t+3}+(S_{t+3}+D_{t+3})a_{t+2}-P_{t+3}c_{t+3}-S_{t+3}a_{t+3}\right]$	…then the period *t*+3 constraint…
$+\ldots$	Infinite number of terms

Infinite-Period Model: Sequential Formulation

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$+\ldots$	Infinite number of terms

- **Step 2: Compute FOCs with respect to c_t, a_t, c_{t+1}, a_{t+1}, c_{t+2}, …**

Identical except for time subscripts

with respect to c_t:	$u'(c_t)-\lambda_t P_t=0$	Equation 1
with respect to a_t:	$-\lambda_t S_t+\beta\lambda_{t+1}(S_{t+1}+D_{t+1})=0$	Equation 2
with respect to c_{t+1}:	$\cancel{\beta}u'(c_{t+1})-\cancel{\beta}\lambda_{t+1}P_{t+1}=0$	Equation 3

The Basics of Asset Pricing

$$u'(c_t) - \lambda_t P_t = 0$$ Equation 1

$$-\lambda_t S_t + \beta\lambda_{t+1}(S_{t+1} + D_{t+1}) = 0$$ Equation 2

$$u'(c_{t+1}) - \lambda_{t+1} P_{t+1} = 0$$ Equation 3

- **Equation 2 →** $$S_t = \left(\frac{\beta\lambda_{t+1}}{\lambda_t}\right)(S_{t+1} + D_{t+1})$$ BASIC ASSET-PRICING EQUATION

Period-*t* stock price = Pricing kernel x Future return

Two components:
1. Future price of stock
2. Future dividend payment

- **Much of finance theory concerned with pricing kernel**
  - **Theoretical properties**
  - **Empirical models of kernels**
- **Pricing kernel where macro theory and finance theory intersect**
- **Why buy an asset?**
  - **May pay a dividend in the future**
  - **Market value may rise in the future: $S_{t+1}/S_t > 1$ is "capital gain"**

Finance Fundamentals

The Basics of Asset Pricing

$$u'(c_t) - \lambda_t P_t = 0$$ Equation 1

$$-\lambda_t S_t + \beta\lambda_{t+1}(S_{t+1} + D_{t+1}) = 0$$ Equation 2

$$u'(c_{t+1}) - \lambda_{t+1} P_{t+1} = 0$$ Equation 3

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- **Much of finance theory concerned with pricing kernel**
  - **Theoretical properties**
  - **Empirical models of kernels**
- **Pricing kernel where macro theory and finance theory intersect**
- **To take more macro-centric view**
  - **Solve equations 1 and 3 for λ_t and λ_{t+1}**
  - **Insert in asset-pricing equation**

MACROECONOMIC EVENTS AFFECT ASSET PRICES

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{P_t}{P_{t+1}}\right)$$

↓ Using definition of inflation: $1+\pi_{t+1} = P_{t+1} / P_t$

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{1}{1+\pi_{t+1}}\right)$$

VIEW AS A CONSUMPTION-SAVINGS OPTIMALITY CONDITION →

- **Consumption across time (c_t and c_{t+1}) affects stock prices**
  - **Fluctuations over time in aggregate consumption impact S_t**
- **Inflation affects stock prices**
  - **Fluctuations over time in inflation impact S_t**
- **ANY factor (monetary policy, fiscal policy, globalization, etc.) that affects inflation and GDP in principle impacts stock/asset markets**
- **Direction of causality?...**

CONSUMER OPTIMIZATION

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{P_t}{P_{t+1}}\right)$$

↓ Move $u'(c_t)$ and $\beta u'(c_{t+1})$ terms to left-hand-side, and S_t to right-hand-side

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \left(\frac{S_{t+1} + D_{t+1}}{S_t}\right)\left(\frac{1}{1+\pi_{t+1}}\right)$$

MRS between period t consumption and period $t+1$ consumption ← i.e., ratio of marginal utilities

Some sort of price ratio…

Consumer Optimization

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{P_t}{P_{t+1}}\right)$$

Move $u'(c_t)$ and $\beta u'(c_{t+1})$ terms to left-hand-side, and S_t to right-hand-side

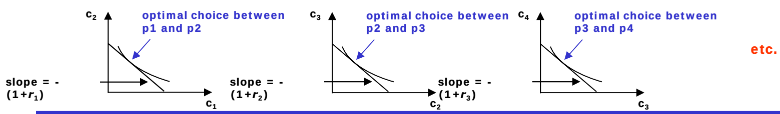
$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \left(\frac{S_{t+1} + D_{t+1}}{S_t}\right)\left(\frac{1}{1+\pi_{t+1}}\right)$$

CONSUMPTION-SAVINGS OPTIMALITY CONDITION

i.e., ratio of marginal utilities → MRS between period t consumption and period $t+1$ consumption

Analogy with Chapters 3 & 4: must be $(1+r_t)$ ← Recall real interest rate is a ***price***

- Recover Chapter 3 & 4 framework by setting t =1 and β = 1
- Infinite-period framework is sequence of overlapping two-period frameworks



A Long-Run Theory of Macro

- Consumption-savings optimality condition at the heart of modern macro analysis
  - Emphasize the dynamic nature of aggregate economic events
  - Foundation for understanding the periodic ups and downs ("business cycles") of the economy
  - (Business cycle analysis after midterm exam)

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_t$$

__NEXT__: Impose "steady state" and examine long-run relationship between interest rates and consumer impatience

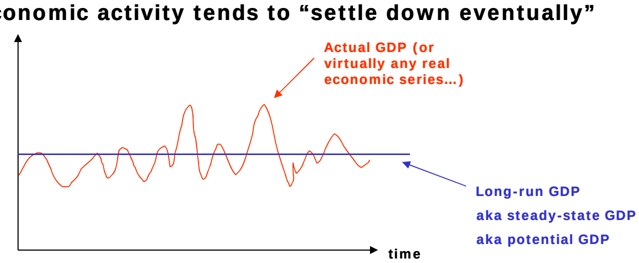
$$\frac{1}{\beta} = 1 + r$$

STEADY-STATE (AKA LONG-RUN): WHY ARE INTEREST RATES POSITIVE?

FEBRUARY 27, 2012

A LONG-RUN THEORY OF MACRO

- Aggregate economic activity tends to "settle down eventually"



- The "ups and downs" are business cycles
- The "average" is the long-run
  - Technical terminology: steady-state
- Business-cycle theory after midterm exam

STEADY STATE

- **Steady state**
  - A concept from differential equations
  - (Optimality conditions of economic models are differential equations…)
  - Heuristic definition: in a dynamic (mathematical) system, a steady-state is a condition in which the variables that are moving over time settle down to constant values
- In dynamic macro models, a steady state is a condition in which all ***real*** variables settle down to constant values
  - But ***nominal*** variables (e.g., price level) may still be fluctuating over time (will be important in monetary analysis – Chapter 14)

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- In dynamic macro models, a steady state is a condition in which all ***real*** variables settle down to constant values
  - But ***nominal*** variables (e.g., price level) may still be fluctuating over time (will be important in monetary analysis – Chapter 14)
  - Simple example
    - Suppose $M_t / P_t = c_t$ is an optimality condition of an economic model (c_t is consumption, P_t is nominal price level, M_t is nominal money stock of economy)
    - Even if c_t eventually becomes constant over time (i.e., reaches a steady-state), it is ***possible*** for ***both*** M_t and P_t to continue growing over time (at the same rate of course…)
- Bottom line: in ss, ***real*** variables do not change over time, nominal variables may change over time (*inflation* is a ***real*** variable)

A LONG-RUN THEORY OF MACRO

- Consumption-savings optimality condition at the heart of modern macro analysis

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_t$$

Steady-state: $c_t = c_{t+1} = c$
and $r_t = r_{t+1} = r$
(i.e., just dropping all time subscripts on real variables!)

$$\frac{u'(c)}{\beta u'(c)} = 1 + r$$

A LONG-RUN THEORY OF MACRO

- Consumption-savings optimality condition at the heart of modern macro analysis

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$$\frac{u'(c)}{\beta u'(c)} = 1 + r$$

$$\frac{1}{\beta} = 1 + r$$

KEY RELATIONSHIP

Inverse of subjective discount factor

(one plus) real interest rate

Real Interest Rate

- **Recall first interpretation of *r***
  - **Price of consumption in a given period in terms of consumption in the next period**
  - **(Chapter 3 & 4: *r* was the price of period-1 consumption in terms of period-2 consumption)**

$$\underbrace{\frac{1}{\beta}}_{\text{Inverse of subjective discount factor}}=\underbrace{1+r}_{\text{(one plus) real interest rate}}$$

- **Long-run (i.e., steady state) real interest rate simply a reflection of degree of impatience of individuals in an economy**
  - **The lower is *β*, the higher is *r***
  - **The more impatient a populace is, the higher are interest rates**

Macro Fundamentals

Real Interest Rate

- **Recall first interpretation of *r***
  - **Price of consumption in a given period in terms of consumption in the next period**
  - **(Chapter 3 & 4: *r* was the price of period-1 consumption in terms of period-2 consumption)**

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- **Long-run (i.e., steady state) real interest rate simply a reflection of degree of impatience of individuals in an economy**
  - **The lower is *β*, the higher is *r***
  - **The more impatient a populace is, the higher are interest rates**

- **Which came first, *β* or *r*?**
  - **Modern macro view: $\beta < 1$ causes $r > 0$, not the other way around**
  - **A deep view of *<u>why</u>* positive interest rates exist in the world**