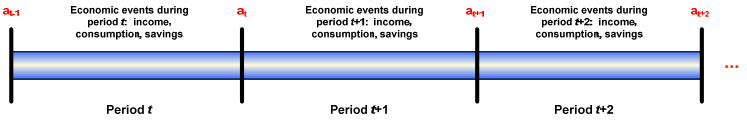
INFINITE-PERIOD CONSUMER ANALYSIS (CONTINUED)

OCTOBER 17, 2011

BASICS

- **Timeline of events**



- **Notation**
  - c_t: consumption in period t
  - P_t: nominal price of consumption in period t
  - Y_t: nominal income in period t ("falls from the sky")
  - a_{t-1}: real wealth (stock) holdings at beginning of period t / end of period t-1

The "defining features" of stock

  - S_t: nominal price of a unit of stock in period t
  - D_t: nominal dividend paid in period t by each unit of stock held at the start of t

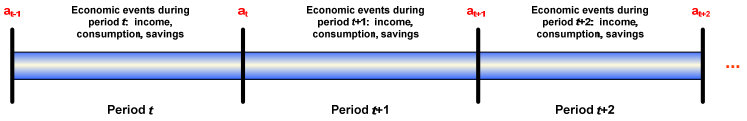
  - π_{t+1}: net inflation rate between period t and period t+1

$$\pi_{t+1} = \frac{P_{t+1} - P_t}{P_t} \left(= \frac{P_{t+1}}{P_t} - 1 \right)$$

  - y_t: real income in period t (= Y_t / P_t)

BASICS

- **Timeline of events**



- **Notation**
  - c_{t+1}: consumption in period $t+1$
  - P_{t+1}: nominal price of consumption in period $t+1$
  - Y_{t+1}: nominal income in period $t+1$ ("falls from the sky")
  - a_t: real wealth (stock) holdings at beginning of period $t+1$ / end of period t

The "defining features" of stock

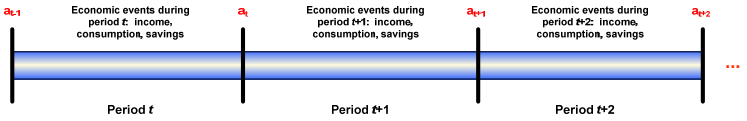
  - S_{t+1}: nominal price of a unit of stock in period $t+1$
  - D_{t+1}: nominal dividend paid in period t by each unit of stock held at the start of $t+1$
  - π_{t+2}: net inflation rate between period $t+1$ and period $t+2$

$$\pi_{t+2} = \frac{P_{t+2} - P_{t+1}}{P_{t+1}} \left(= \frac{P_{t+2}}{P_{t+1}} - 1 \right)$$

  - y_{t+1}: real income in period $t+1$ ($= Y_{t+1} / P_{t+1}$)

BASICS

- **Timeline of events**



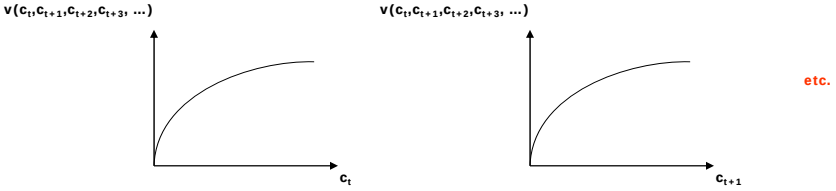
- **Notation**
  - And so on for period $t+2$, $t+3$, etc...

SUBJECTIVE DISCOUNT FACTOR

- Infinite number of periods a more serious view of time
- Impatience potentially an issue when taking a serious view of time
- Individuals (i.e., consumers) are impatient
  - All else equal, would rather have X utils today than identical X utils at some future date
  - An introspective statement about the world
  - An empirical statement about the world
- Subjective discount factor
  - A simple model of consumer impatience
  - β (a number between zero and one) measures impatience
    - The lower is β, the less does individual value future utility
  - Simple assumption about how "impatience" builds up over time
    - Multiplicatively: i.e., discount one period ahead by β, discount two periods ahead by β^2, discount three periods ahead by β^3, etc.
    - Do individuals' impatience really build up over time in this way?...limited empirical evidence so really don't know…

UTILITY

- Preferences $v(c_t, c_{t+1}, c_{t+2}, \ldots)$ with all the "usual properties"
  - Lifetime utility function
  - Strictly increasing in c_t, c_{t+1}, c_{t+2}, c_{t+3}, …
  - Diminishing marginal utility in c_t, c_{t+1}, c_{t+2}, c_{t+3}, …



- Lifetime utility function additively-separable across time (a simplifying assumption), starting at time t

$$v(c_t, c_{t+1}, c_{t+2}, c_{t+3}, \ldots) = u(c_t) + \beta u(c_{t+1}) + \beta^2 u(c_{t+2}) + \beta^3 u(c_{t+3}) + \ldots$$

- Utility side of infinite-period framework no different than Chapter 1 model – except not as simple to represent graphically

BUDGET CONSTRAINT(S)

- Suppose again Y "falls from the sky"
  - Y_t in period t, Y_{t+1} in period $t+1$, Y_{t+2} in period $t+2$, etc.
- Need infinite budget constraints to describe economic opportunities and possibilities
  - One for each period
  - Period-t budget constraint

$$\underbrace{P_t c_t + S_t a_t}_{\text{Total expenditure in period } t} = \underbrace{Y_t + S_t a_{t-1} + D_t a_{t-1}}_{\text{Total income in period } t}$$

Total expenditure in period t: period-t consumption + wealth to *carry into period* $t+1$

Total income in period t: period-t Y + income from stock-holdings *carried into period* t (has value S_t and pays dividend D_t)

  - Period $t+1$ budget constraint

$$\underbrace{P_{t+1} c_{t+1} + S_{t+1} a_{t+1}}_{\text{Total expenditure in period } t+1} = \underbrace{Y_{t+1} + S_{t+1} a_t + D_{t+1} a_t}_{\text{Total income in period } t+1}$$

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Savings during period t (a flow); Dividend income during period t (a flow)

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Savings during period $t+1$ (a flow); Dividend income during period $t+1$ (a flow)

Total expenditure in period $t+1$: period-$t+1$ consumption + wealth to *carry into period* $t+2$

Total income in period $t+1$: period-$t+1$ Y + income from stock-holdings *carried into period* $t+1$ (has value S_{t+1} and pays dividend D_{t+1})

And identical-looking budget constraints for $t+2$, $t+3$, $t+4$, etc...

LAGRANGE ANALYSIS: SEQUENTIAL APPROACH

❑ **Step 1: Construct Lagrange function (starting from *t*)**

IMPORTANT: Discount factor β multiplies ***both*** future utility ***and*** future budget constraints

Everything (utility ***and*** income) about the future is discounted

$$u(c_t) + \beta u(c_{t+1}) + \beta^2 u(c_{t+2}) + \beta^3 u(c_{t+3}) + \ldots$$ — First the lifetime utility function....

$$+\lambda_t \left[Y_t + (S_t + D_t)a_{t-1} - P_t c_t - S_t a_t \right]$$ — ...then the period *t* constraint...

$$+\beta\lambda_{t+1} \left[Y_{t+1} + (S_{t+1} + D_{t+1})a_t - P_{t+1} c_{t+1} - S_{t+1} a_{t+1} \right]$$ — ...then the period *t*+1 constraint...

$$+\beta^2\lambda_{t+2} \left[Y_{t+2} + (S_{t+2} + D_{t+2})a_{t+1} - P_{t+2} c_{t+2} - S_{t+2} a_{t+2} \right]$$ — ...then the period *t*+2 constraint...

$$+\beta^3\lambda_{t+3} \left[Y_{t+3} + (S_{t+3} + D_{t+3})a_{t+2} - P_{t+3} c_{t+3} - S_{t+3} a_{t+3} \right]$$ — ...then the period *t*+3 constraint...

$$+\ldots$$ — Infinite number of terms

Infinite-Period Model: Sequential Formulation

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$$+\ldots$$ — Infinite number of terms

❑ **Step 2: Compute FOCs with respect to c_t, a_t, c_{t+1}, a_{t+1}, c_{t+2}, ...**

Identical except for time subscripts

- with respect to c_t: $u'(c_t) - \lambda_t P_t = 0$ — Equation 1
- with respect to a_t: $-\lambda_t S_t + \beta\lambda_{t+1}(S_{t+1} + D_{t+1}) = 0$ — Equation 2
- with respect to c_{t+1}: $\beta u'(c_{t+1}) - \beta\lambda_{t+1} P_{t+1} = 0$ — Equation 3

THE BASICS OF ASSET PRICING

$$u'(c_t) - \lambda_t P_t = 0 \quad \text{Equation 1}$$
$$-\lambda_t S_t + \beta\lambda_{t+1}(S_{t+1} + D_{t+1}) = 0 \quad \text{Equation 2}$$
$$u'(c_{t+1}) - \lambda_{t+1} P_{t+1} = 0 \quad \text{Equation 3}$$

- **Equation 2 →** $S_t = \left(\frac{\beta\lambda_{t+1}}{\lambda_t}\right)(S_{t+1} + D_{t+1})$ **BASIC ASSET-PRICING EQUATION**

Period-*t* stock price = Pricing kernel

- **Much of finance theory concerned with pricing kernel**
  - **Theoretical properties**
  - **Empirical models of kernels**
- **Pricing kernel where macro theory and finance theory intersect**
  - **Allows studying common "macro factors" that affect "all" asset markets / asset prices**
  - **Chapter 14: link between pricing kernel and bond prices**

THE BASICS OF ASSET PRICING

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Period-*t* stock price = Pricing kernel × Future return

Two components:
1. Future price of stock
2. Future dividend payment

- **Much of finance theory concerned with pricing kernel**
  - **Theoretical properties**
  - **Empirical models of kernels**
- **Pricing kernel where macro theory and finance theory intersect**
- **Why buy an asset?**
  - **May pay a dividend in the future**
  - **Market value may rise in the future: $S_{t+1}/S_t > 1$ is "capital gain"**

THE BASICS OF ASSET PRICING

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- Much of finance theory concerned with pricing kernel
  - Theoretical properties
  - Empirical models of kernels
- Pricing kernel where macro theory and finance theory intersect
- To take more macro-centric view
  - Solve equations 1 and 3 for λ_t and λ_{t+1}
  - Insert in asset-pricing equation

MACROECONOMIC EVENTS AFFECT ASSET PRICES

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{P_t}{P_{t+1}}\right)$$

Using definition of inflation: $1+\pi_{t+1} = P_{t+1} / P_t$

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{1}{1+\pi_{t+1}}\right)$$

→ VIEW AS A CONSUMPTION-SAVINGS OPTIMALITY CONDITION

- Consumption across time (c_t and c_{t+1}) affects stock prices
  - Fluctuations over time in aggregate consumption impact S_t
- Inflation affects stock prices
  - Fluctuations over time in inflation impact S_t
- ANY factor (monetary policy, fiscal policy, globalization, etc.) that affects inflation and GDP in principle impacts stock / asset markets
- Direction of causality?...

CONSUMER OPTIMIZATION

$$S_t = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)(S_{t+1} + D_{t+1})\left(\frac{P_t}{P_{t+1}}\right)$$

Move $u'(c_t)$ and $\beta u'(c_{t+1})$ terms to left-hand-side, and S_t to right-hand-side

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \left(\frac{S_{t+1} + D_{t+1}}{S_t}\right)\left(\frac{1}{1+\pi_{t+1}}\right)$$

i.e., ratio of marginal utilities → MRS between period t consumption and period $t+1$ consumption

Some sort of price ratio...

CONSUMER OPTIMIZATION

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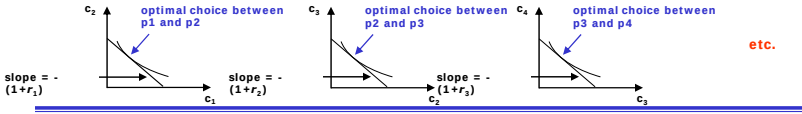
CONSUMPTION-SAVINGS OPTIMALITY CONDITION

i.e., ratio of marginal utilities → MRS between period t consumption and period $t+1$ consumption

Analogy with Chapters 3 & 4: must be $(1+r_t)$ ← Recall real interest rate is a *price*

- Recover Chapter 3 & 4 framework by setting $t = 1$ and $\beta = 1$
- Infinite-period framework is sequence of overlapping two-period frameworks



A LONG-RUN THEORY OF MACRO

- Consumption-savings optimality condition at the heart of modern macro analysis
  - Emphasize the dynamic nature of aggregate economic events
  - Foundation for understanding the periodic ups and downs ("business cycles") of the economy
  - (Business cycle analysis after midterm exam)

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_t$$

NEXT: Impose "steady state" and examine long-run relationship between interest rates and consumer impatience

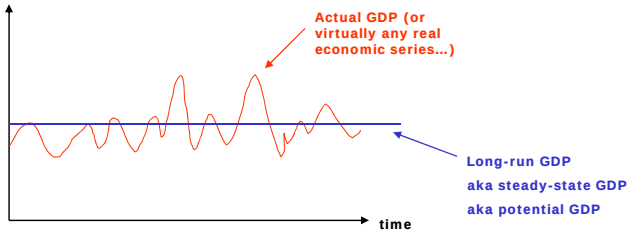
$$\frac{1}{\beta} = 1 + r$$

STEADY-STATE (LONG-RUN) OF INFINITE-PERIOD ECONOMY: WHY ARE INTEREST RATES POSITIVE?

OCTOBER 17, 2011

A Long-Run Theory of Macro

- Aggregate economic activity tends to "settle down eventually"



  - The "ups and downs" are business cycles
  - The "average" is the long-run
    - Technical terminology: steady-state
  - Business-cycle theory after midterm exam

Steady State

- Steady state
  - A concept from differential equations
  - (Optimality conditions of economic models are differential equations…)
  - Heuristic definition: in a dynamic (mathematical) system, a steady-state is a condition in which the variables that are moving over time settle down to constant values
- In dynamic macro models, a steady state is a condition in which all *real* variables settle down to constant values
  - But *nominal* variables (e.g., price level) may still be fluctuating over time (will be important in monetary analysis – Chapter 14)

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- In dynamic macro models, a steady state is a condition in which all *real* variables settle down to constant values
  - But *nominal* variables (e.g., price level) may still be fluctuating over time (will be important in monetary analysis – Chapter 14)
  - Simple example
    - Suppose $M_t/P_t = c_t$ is an optimality condition of an economic model (c_t is consumption, P_t is nominal price level, M_t is nominal money stock of economy)
    - Even if c_t eventually becomes constant over time (i.e., reaches a steady-state), it is *possible* for *both* M_t and P_t to continue growing over time (at the same rate of course…)

- Bottom line: in ss, *real* variables do not change over time, nominal variables may change over time (*inflation* is a *real* variable)

A LONG-RUN THEORY OF MACRO

- Consumption-savings optimality condition at the heart of modern macro analysis

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_t$$

Steady-state: $c_t = c_{t+1} = c$ and $r_t = r_{t+1} = r$ (i.e., just dropping all time subscripts on real variables!)

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$$\frac{1}{\beta} = 1 + r$$

Inverse of subjective discount factor; (one plus) real interest rate

KEY RELATIONSHIP

Real Interest Rate

- Recall first interpretation of *r*
  - Price of consumption in a given period in terms of consumption in the next period
  - (Chapter 3 & 4: *r* was the price of period-1 consumption in terms of period-2 consumption)

$$\frac{1}{\beta} = 1 + r$$

Inverse of subjective discount factor; (one plus) real interest rate

- Long-run (i.e., steady state) real interest rate simply a reflection of degree of impatience of individuals in an economy
  - The lower is β, the higher is *r*
  - The more impatient a populace is, the higher are interest rates

REAL INTEREST RATE

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$$\underbrace{\frac{1}{\beta}}_{\text{Inverse of subjective discount factor}} = \underbrace{1+r}_{\text{(one plus) real interest rate}}$$

- Long-run (i.e., steady state) real interest rate simply a reflection of degree of impatience of individuals in an economy
  - The lower is β, the higher is r
  - The more impatient a populace is, the higher are interest rates

- Which came first, β or r?
  - Modern macro view: $\beta < 1$ causes $r > 0$, not the other way around
  - A deep view of ***why*** positive interest rates exist in the world