Economics 602

Macroeconomic Theory and Policy

Problem Set 1 Suggested Solutions

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1. **Partial Derivatives.** For each of the following multi-variable functions, compute the partial derivatives with respect to both *x* and *y*.

   Solution: In order to compute the partial derivative with respect to *x*, momentarily pretend that *y* is a constant (for example, imagine momentarily that *y* = 5) and proceed to differentiate using the usual rules of calculus. Likewise, in order to compute the partial derivative with respect to *y*, momentarily pretend that *x* is a constant (for example, imagine momentarily that *x* = 5) and proceed to differentiate using the usual rules of calculus.

   Applying this algorithm to each of the given functions:

   a. $f(x,y) = xy$

   We have $f_x(x,y) = y$ and $f_y(x,y) = x$.

   b. $f(x,y) = 2x + 3y$

   We have $f_x(x,y) = 2$ and $f_y(x,y) = 3$.

   c. $f(x,y) = x^2 y^4$

   We have $f_x(x,y) = 2xy^4$ and $f_y(x,y) = 4x^2 y^3$.

   d. $f(x,y) = \ln x + 2\ln y$

   We have $f_x(x,y) = 1/x$ and $f_y(x,y) = 2/y$.

   e. $f(x,y) = 2\sqrt{x} + 2\sqrt{y}$

   Recall from principles of basic mathematics that we can write this function as $f(x,y) = 2x^{1/2} + 2y^{1/2}$. Hence, the partial derivatives are $f_x(x,y) = x^{-1/2} = 1/\sqrt{x}$ and $f_y(x,y) = y^{-1/2} = 1/\sqrt{y}$.

f. $f(x,y) = \dfrac{x}{y}$

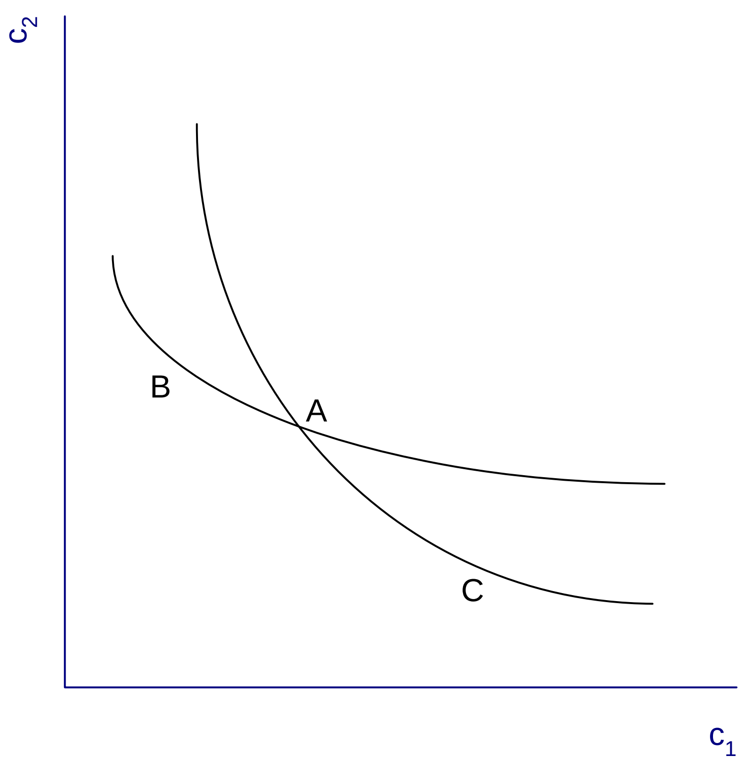
Recall from principles of basic mathematics that we can write this function as $f(x,y) = xy^{-1}$. Hence, the partial derivatives are $f_x(x,y) = y^{-1} = 1/y$ and $f_y(x,y) = -xy^{-2} = -x/y^2$.

g. $f(x,y) = \dfrac{y}{x}$

Recall from principles of basic mathematics that we can write this function as $f(x,y) = yx^{-1}$. Hence, the partial derivatives are $f_x(x,y) = -yx^{-2} = -y/x^2$ and $f_y(x,y) = x^{-1} = 1/x$

2. **Properties of Indifference Maps.** For the general model of utility functions and indifference maps developed in class, explain why no two indifference curves can ever cross each other. Your answer must explain the economic logic here, and may also include appropriate equations and/or graphs.

Solution: The proof proceeds by contradiction. Consider the following indifference curves that cross each other.



The consumption bundle A lies on both indifference curves. Because bundle A and bundle B lie on the same indifference curve, they yield the same level of utility. Likewise, because bundle A and bundle C lie on the same indifference curve, they must yield the same level of utility. This then implies that bundle B and bundle C yield the same level of utility (transitive property of preferences). But if this were true, then bundle B and bundle C should lie on the same indifference curve, which they do not by assumption. Thus, we have reached a logical contradiction – indifference curves cannot cross each other.

3. **A Canonical Utility Function.** Consider the utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma},$$

where c denotes consumption of some arbitrary good and σ (the Greek letter "sigma") is known as the "curvature parameter" because its value governs how curved the utility function is. In the following, restrict your attention to the region $c > 0$ (because "negative consumption" is an ill-defined concept). The parameter σ is treated as a constant.

a. Plot the utility function for $\sigma = 0$. Does this utility function display diminishing marginal utility? Is marginal utility ever negative for this utility function?

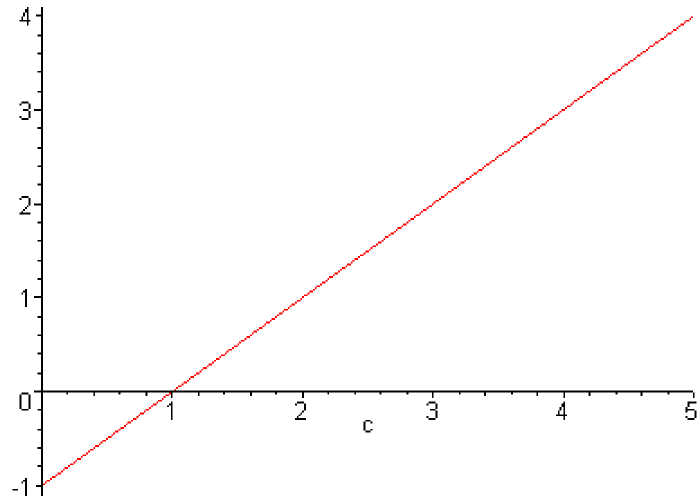
b. Plot the utility function for $\sigma = 1/2$. Does this utility function display diminishing marginal utility? Is marginal utility ever negative for this utility function?

c. Consider instead the natural-log utility function $u(c) = \ln(c)$. Does this utility function display diminishing marginal utility? Is marginal utility ever negative for this utility function?

d. Determine the value of σ (if any value exists at all) that makes the general utility function presented above collapse to the natural-log utility function in part c. (**Hint:** Examine the derivatives of the two functions.)

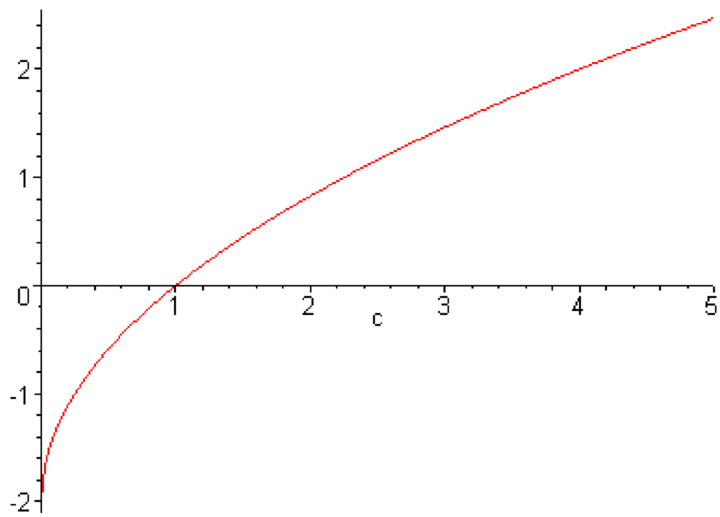
Solution:

a. With $\sigma = 0$, the utility function becomes the linear function $u(c) = c - 1$, which has a simple graph:



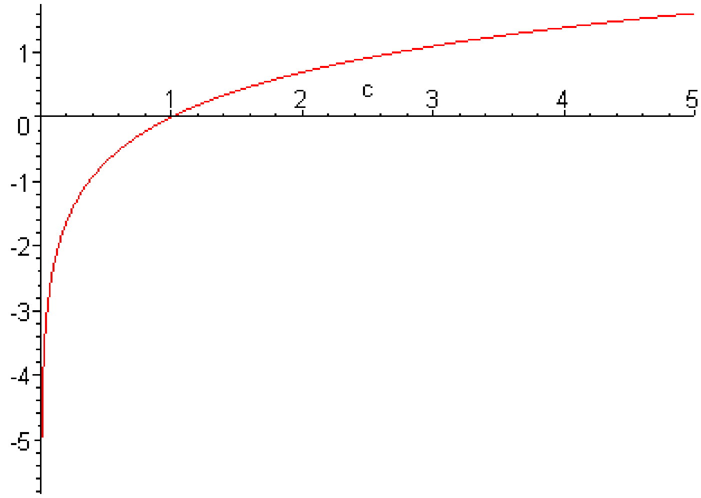
Notice that utility may actually be negative – but recall that the units of utility (utils) are completely arbitrary, so there is nothing wrong with considering negative values of utility. This linear function clearly does **not** display diminishing marginal utility because its slope is constant at one throughout – so of course, marginal utility never becomes negative either.

b. With $\sigma = 1/2$, the utility function becomes $u(c) = 2\sqrt{c} - 2$, which looks like



Notice again that here we have negative values of utility, which is fine since utility is measured in an arbitrary scale. The slope of this utility function is $u'(c) = 1/\sqrt{c}$ (recall the slope of a function is simply the first derivative), which is always positive as long as consumption is positive, so marginal utility is never negative. And clearly as consumption rises, the slope falls, so this function does display diminishing marginal utility.

c. The natural log utility function has graph



The derivative of this function is $u'(c) = 1/c$, which is always positive when consumption is positive, so this function also does not ever experience negative marginal utility. As consumption rises, the slope of the utility function falls, so this function does display diminishing marginal utility.

d. The derivative of the log utility function is $u'(c) = 1/c$, while the derivative of the general utility function presented above is $u'(c) = c^{-\sigma}$. Clearly, for $\sigma = 1$ the slopes of the two functions (that is, the two marginal utility functions) are identical. This does **not** prove that the two functions are identical, however, because it could be that the two functions always have the same slope but are vertical translates of each other. To see that they are indeed the same function, however, try plotting the general utility function for several values of the curvature parameter near one (that is, try plotting the function for, say, $\sigma = 0.9, \sigma = 0.95, \sigma = 0.99, \sigma = 1.05, \sigma = 1.01$, etc) – you will see that as the curvature parameter approaches one from either direction, the general utility function approaches the log utility function.

4. **The Implicit Function Theorem and the Marginal Rate of Substitution.** An important result from multivariable calculus is the **implicit function theorem,** which states that given a function $f(x, y)$, the derivative of y with respect to x is given by

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y},$$

where $\partial f / \partial x$ denotes the partial derivative of f with respect to x and $\partial f / \partial y$ denotes the partial derivative of f with respect to y. Simply stated, a partial derivative of a multivariable function is the derivative of that function with respect to one particular variable, treating all other variables as constant. For example, suppose $f(x, y) = xy^2$. To compute the partial derivative of f with respect to x, we treat y as a constant, in which case we obtain $\partial f / \partial x = y^2$, and to compute the partial derivative of f with respect to y, we treat x as a constant, in which case we obtain $\partial f / \partial y = 2xy$.

We have described the slope of an indifference curve as the marginal rate of substitution between the two goods. Imagining that c_2 is plotted on the vertical axis and c_1 plotted on the horizontal axis, compute the marginal rate of substitution for the following utility functions.

a. $u(c_1, c_2) = \ln(c_1) + \ln(c_2)$
b. $u(c_1, c_2) = \sqrt{c_1} + \sqrt{c_2}$
c. $u(c_1, c_2) = c_1^a c_2^{1-a}$, where $a \in (0,1)$ is some constant.

Solution: Simply use the implicit function theorem in each case to compute the slope of the indifference curves (which is the negative of the marginal rate of substitution):

a. With the given utility function, we have $\partial u / \partial c_1 = 1 / c_1$ and $\partial u / \partial c_2 = 1 / c_2$, so that, by the implicit function theorem, the slope of each indifference curve is

$$\frac{dc_2}{dc_1} = -\frac{1/c_1}{1/c_2} = -\frac{c_2}{c_1}$$

b. With the given utility function, we have that $\partial u / \partial c_1 = 0.5 / \sqrt{c_1}$ and $\partial u / \partial c_2 = 0.5\sqrt{c_2}$, so that, by the implicit function theorem, the slope of each indifference curve is

$$\frac{dc_2}{dc_1} = -\frac{0.5/\sqrt{c_1}}{0.5/\sqrt{c_2}} = -\sqrt{\frac{c_2}{c_1}}$$

c. With the given utility function, we have that $\partial u / \partial c_1 = a c_1^{a-1} c_2^{1-a}$ and $\partial u / \partial c_2 = (1-a) c_1^a c_2^{-a}$, so that, by the implicit function theorem, the slope of each indifference curve is

$$\frac{dc_2}{dc_1} = -\frac{ac_1^{a-1}c_2^{1-a}}{(1-a)c_1^a c_2^{-a}} = -\left(\frac{a}{1-a}\right)\left(\frac{c_2}{c_1}\right)$$

Notice that in each case, the negative of the slope is a decreasing function of c_1, which graphically is simply our statement that indifference curves generally become flatter the further along the horizontal axis we travel (that is, indifference curves are convex to the origin).

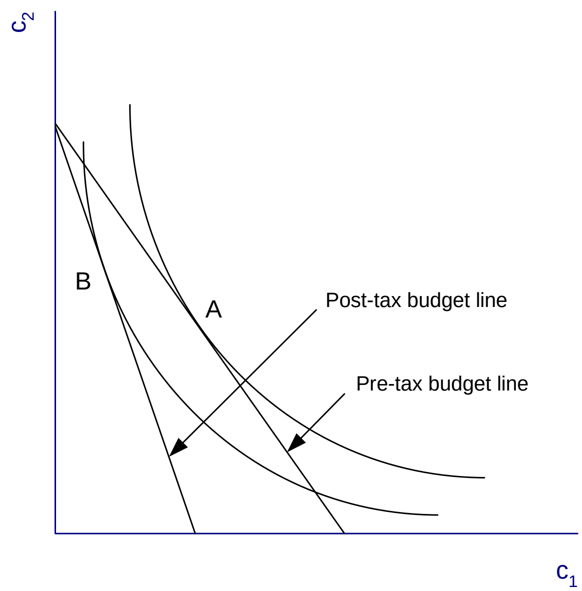
5. **Sales Tax.** Consider the standard consumer problem we have been studying, in which a consumer has to choose consumption of two goods c_1 and c_2 which have prices (in terms of money) P_1 and P_2, respectively. These prices are prices before any applicable taxes. Many states charge sales tax on some goods but not on others – for example, many states charge sales tax on all goods except food and clothes. Suppose that good 1 carries a per-unit sales tax, while good 2 has no sales tax. Use the variable t_1 to denote this sales tax, where t_1 is a number between zero and one (so, for example, if the sales tax on good 1 were 15 percent, we would have $t_1 = 0.15$).
   a. With sales tax t_1 and consumer income Y, write down the budget constraint of the consumer. Explain economically how/why this budget constraint differs from the standard one we have been considering thus far.
   b. Graphically describe how the imposition of the sales tax on good 1 alters the optimal consumption choice (ie, how the optimal choice of each good is affected by a policy shift from $t_1 = 0$ to $t_1 > 0$).
   c. Suppose the consumer's utility function is given by $u(c_1, c_2) = \log c_1 + \log c_2$. Using a Lagrangian, solve algebraically for the consumer's optimal choice of c_1 and c_2 as functions of P_1, P_2, t_1, and Y. Graphically show how, for this particular utility function, the optimal choice changes due to the imposition of the sales tax on good 1.

Solution:

a. The sales tax on good 1 is paid in addition to its price P_1. Thus, the total expenditure on good 1 (i.e., the total amount of money the consumer will pay out of his pocket to purchase good 1) is $(1+t_1)P_1c_1$. For example, in Massachusetts the sales tax rate is 5 percent, so we would have $t_1 = 0.05$. This means that you would pay 105 percent ($1+t_1 = 1.05$) of the pre-tax price. Because there is no tax on good 2, the budget constraint is $(1+t_1)P_1c_1 + P_2c_2 = Y$.
b. If we solve the budget constraint for c_2 in terms of c_1, we get

$$c_2 = -\frac{P_1(1+t_1)}{P_2}c_1 + \frac{Y}{P_2},$$

so clearly the slope of the budget line is affected. **This is an important general lesson: the reason why taxes levied on consumers have (or do not have) effects is because in general they alter the budget constraint.** Intuitively, if the consumer now spent all of his income on good 1, he would be able to buy less than without the sales tax, but if spent all of his income on good 2, the quantity of good 2 he could buy would be unaffected by the imposition of the sales tax on good 1. Graphically, the budget line becomes steeper by pivoting around the vertical intercept, as shown in the figure below. Because the consumer's optimal choice of (c_1, c_2) is described by the tangency of an indifference curve with the budget line, the figure also shows how the optimal choice changes – in this case, the optimal choice moves from point A to point B. As drawn in the figure, the new optimal choice features less consumption of good 1 and more consumption of good 2 – the consumer has substituted some good 2 for some good 1 in the face of a rise in price (**inclusive of tax)** of good 1. **But see part c below for more on this substitution effect.**



After the imposition of the sales tax on good 1, the optimal choice moves from point A to point B, which features less consumption of good 1.

c. The consumer's problem is to maximize $\ln c_1 + \ln c_2$ subject to the budget constraint $Y - P_1 c_1 - P_2 c_2 = 0$. The Lagrangian for this problem is

$$L(c_1, c_2, \lambda) = \ln c_1 + \ln c_2 + \lambda(Y - (1+t_1)P_1 c_1 - P_2 c_2),$$

where λ denotes the Lagrange multiplier on the budget constraint. The first-order conditions of L with respect to c_1, c_2, and λ are, respectively,

$$\frac{1}{c_1} - \lambda(1+t_1)P_1 = 0$$
$$\frac{1}{c_2} - \lambda P_2 = 0$$
$$Y - (1+t_1)P_1c_1 - P_2c_2 = 0$$

We must solve this system of equations for c_1 and c_2. There are of course many ways to solve this system, which only differ in the exact order of equations used. One useful way of proceeding is to first eliminate the multiplier λ. To do this, from the second equation, we get $\lambda = \frac{1}{P_2c_2}$. Substituting this into the first equation gives

$$\frac{1}{c_1} - \frac{P_1(1+t_1)}{P_2c_2} = 0.$$

From here, we can solve for P_2c_2, which is simply total expenditure on good 2:

$$P_2c_2 = (1+t_1)P_1c_1$$

This expression states that total expenditure on good 2 equals total expenditure on good 1 (inclusive of the tax on good 1). **Note that this result need not always hold it holds here because of the given utility function.**

Proceeding, substitute the above expression into the budget constraint to get

$$Y - (1+t_1)P_1c_1 - (1+t_1)P_1c_1 = 0,$$

from which it easily follows that the optimal choice of consumption of good 1 is

$$c_1^* = \frac{Y}{2(1+t_1)P_1}.$$

If we had numerical values for the objects on the right-hand-side, clearly we would know numerically the optimal choice of consumption of good 1. This solution reveals that a rise in t_1 (holding constant Y and P_1) results in a fall in c_1^*.

To obtain the optimal choice of consumption of good 2, return to the above expression $P_2c_2 = (1+t_1)P_1c_1$, from which we get that consumption of good 2 is related to consumption of good 1 in the following way: $c_2 = \left(\frac{(1+t_1)P_1}{P_2}\right)c_1$. Inserting the optimal choice of good 1 found above, we have that the optimal choice of good 2 is

$$c_2^* = \frac{Y}{2P_2}.$$

Thus, even though the consumer splits his income evenly on expenditures on good 1 and good 2, consumption of good 1 and good 2 are not the same unless $t_1 = 0$. **But notice from the above expression that the optimal choice of good 2 is independent of the tax rate on good 1!** Thus, if we plot this utility function and the associated optimal choices as in part b above, we would get a movement of the optimal choice **straight left from point A**, the original optimal choice, meaning the optimal choice of good 2 is identical while the optimal choice of good 1 decreases.