Economics 602

Macroeconomic Theory and Policy

Problem Set 4 Suggested Solutions

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1. **Optimal Choice in the Consumption-Savings Model with Credit Constraints: A Numerical Analysis.** Consider our usual two-period consumption-savings model. Let preferences of the representative consumer be described by the utility function

$$u(c_1, c_2) = \sqrt{c_1} + \beta\sqrt{c_2},$$

where c_1 denotes consumption in period one and c_2 denotes consumption in period two. The parameter β is known as the subjective discount factor and measures the consumer's degree of impatience in the sense that the smaller is β, the higher the weight the consumer assigns to present consumption relative to future consumption. Assume that $\beta = 1/1.1$. For this particular utility specification, the marginal utility functions are given by $u_1(c_1, c_2) = \dfrac{1}{2\sqrt{c_1}}$ and $u_2(c_1, c_2) = \dfrac{\beta}{2\sqrt{c_2}}$.

The representative household has initial **real** financial wealth (including interest) of $a_0 = 1$ The household earns $y_1 = 5$ units of goods in period one and $y_2 = 10$ units in period two. The real interest rate paid on assets held from period one to period two equals 10% (i.e., $r_1 = 0.1$).

a. Calculate the equilibrium levels of consumption in periods one and two. (**Hint:** Set up the Lagrangian and solve.)
b. Suppose now that lenders to this consumer impose **credit constraints** on the consumer. Specifically, they impose the tightest possible credit constraint – the consumer is not allowed to be in debt at the end of period one, which implies that the consumer's real wealth at the end of period one must be nonnegative ($a_1 \geq 0$) (**Note:** here, a_1 is defined as being exclusive of interest, in contrast to the definition of a_0 above). What is the consumer's choice of period-one and period-two consumption under this credit constraint? Briefly explain, either logically or graphically or both.
c. Does the credit constraint described in part b enhance or diminish welfare (i.e., does it increase or decrease lifetime utility)? Specifically, find the level of utility under the credit constraint and compare it to the level of utility obtained under no credit constraint.

Suppose now that the consumer experiences a temporary increase in real income in period one to $y_1 = 9$, with real income in period two unchanged.

d. Calculate the effect of this positive surprise in income on c_1 and c_2, supposing that there is no credit constraint on the consumer.
e. Finally, suppose that the credit constraint described in part b is back in place. Will it be binding? That is, will it affect the consumer's choices?

Solution:

a. The consumer's problem is to maximize lifetime utility (given by $u(c_1, c_2)$ subject to the LBC. The Lagrangian for this problem is thus

$$L(c_1, c_2, \lambda) = u(c_1, c_2) + \lambda\left(a_0 + y_1 + \frac{y_2}{1+r_1} - c_1 - \frac{c_2}{1+r_1}\right),$$

where we must include the nonzero initial real wealth a_0. The first-order conditions with respect to c_1 and c_2 are

$$u_1(c_1, c_2) - \lambda = 0$$

$$u_2(c_1, c_2) - \frac{\lambda}{1+r_1} = 0$$

Combining these, we get the usual consumption-savings optimality condition, $u_1(c_1, c_2) = (1+r_1)u_2(c_1, c_2)$ (ie, the MRS equals the slope of the LBC). Using the the given utility function, at the optimal choice the following condition must be satisfied:

$$\frac{1}{2\sqrt{c_1}} = (1+r_1)\frac{\beta}{2\sqrt{c_2}}.$$

Solving this expression for c_2 as a function of c_1 gives $c_2 = (1+r_1)^2\beta^2 c_1$. With the specific values given, this turns out to be $c_2 = c_1$. Substituting this into the lifetime budget constraint then yields

$$c_1 + \frac{c_1}{1+r_1} = a_0 + y_1 + \frac{y_2}{1+r_1}.$$

Solving for c_1 gives

$$c_1 = \left[\frac{1+r_1}{2+r_1}\right]\left[a_0 + y_1 + \frac{y_2}{1+r_1}\right],$$

which, when using the values provided, yields $c_1 = 7.90$ and hence $c_2 = 7.90$. **Note also** that, although you were not asked to compute it, you could find the implied value for a_1 using the period one budget constraint $c_1 + a_1 = a_0 + y_1$. This yields that $a_1 = -1.9$, indicating that the household chooses to be a debtor at the end of period one.

b. The imposition of these credit constraints **will be binding** on the consumer's behavior. That is, it will alter the choices made by the household, as can be seen from the fact that in the absence of the credit constraints in part a, the consumer chose to be in debt at the end of period one. Now, being restricted to hold a nonnegative asset position at the end of period one, it will choose that asset position closest to its unrestricted choice but which also satisfies the credit constraint – that is, the consumer will choose $a_1 = 0$. The period one budget constraint, $c_1 + a_1 = a_0 + y_1$ then implies that $c_1 = 6$. The household will simply consume all it can in period one, which is the sum of its endowment and initial assets (inclusive of interest income on those initial assets). It remains now to solve for c_2. Examining the period two budget constraint, $c_2 + a_2 = (1+r_1)a_1 + y_2$ with the condition $a_2 = 0$ imposed and $B_1^* = 0$ shows that $c_2 = y_2 = 10$. **Extension:** At this credit-constrained choice of consumption, the MRS clearly does not equal the slope of the LBC. The slope of the LBC is the market interest rate $1+r$, as usual. However, we can define an "effective interest rate" for this consumer, which is the interest rate that **would** need to prevail for the choice $c_1 = 6, c_2 = 10$ to be the **unrestricted** optimal choice. We can obtain this from the condition $u_1(c_1, c_2) = (1+r)u_2(c_1, c_2)$. This condition is the same as where we started question 1a with, except now, knowing values for c_1 and c_2, we will use it to determine the consumer's effective interest rate. Plugging the values $c_1 = 6, c_2 = 10$ into this condition and solving for the interest rate gives us $r = 0.42$ as the effective interest rate, the interest rate that would have made this choice the consumer's unrestricted optimal choice.

c. With the values for consumption in each of the two periods from parts a and b, the utility function shows that utility without credit constraints equals $u(c_1, c_2) = 5.34$ and utility with credit constraints is $u(c_1, c_2) = 5.29$. Utility is lower under credit constraints, thus welfare is reduced by their imposition. This should strike you as sensible – the consumer wanted (rationally and with perfect information) to be in debt at the end of period one, but banks were unwilling to lend, thus the consumer is worse off. Graphically, this means that the chosen consumption bundle under credit constraints lies on an indifference curve lower than the chosen consumption bundle in the absence of credit constraints. (Technical note: You cannot say something like, "welfare is not lowered by much" because of credit constraints. Although we did not discuss it, the numbers attached to the utility function themselves have no economic meaning – all they are used for is comparing relative welfare, not for making any absolute statements about welfare. You are not responsible for knowing this technical detail, but FYI.)

d. Using exactly the same solution procedure as in part a, you get that $c_1 = 10$ and $c_2 = 10$. Implied by this choice of consumption is that $a_1 = 0$ (using the period one budget constraint). That is, the optimal choice of the consumer following the positive income shock involves a zero asset position at the end of period one.

e. With the credit constraint now back in place (with $y_1 = 9$), there will be no change in household behavior relative to the case without the credit constraint. That is, in part d, the optimal choice of households already involves a choice for

a_1 that satisfies the credit constraint. Thus, the credit constraint is not binding, and welfare is unaffected.

2. **Government in the Two-Period Economy.** Consider again our usual two-period consumption-savings model, augmented with a government sector. Each consumer has preferences described by the utility function

$$u(c_1, c_2) = \ln c_1 + \ln c_2,$$

where ln stands for the natural logarithm, c_1 is consumption in period one, and c_2 is consumption in period two. The associated marginal utility functions are

$$u_1(c_1, c_2) = \frac{1}{c_1} \text{ and } u_2(c_1, c_2) = \frac{1}{c_2}.$$

Suppose that both households and the government start with zero initial assets (i.e., $A_0 = 0$ and $b_0 = 0$), and that the real interest rate is always 10 percent. Assume that government purchases in the first period are one ($g_1 = 1$) and in the second period are 9.9 ($g_2 = 9.9$). In the first period, the government levies lump-sum taxes in the amount of 8 ($t_1 = 8$). Finally, the real incomes of the consumer in the two periods are $y_1 = 9$ and $y_2 = 23.1$.

a. What are lump-sum taxes in period two (t_2), given the above information?
b. Compute the optimal level of consumption in periods one and two, as well as national savings in period one.
c. Consider a tax cut in the first period of 1 unit, with government purchases left unchanged. What is the change in national savings in period one? Provide intuition for the result you obtain.
d. Now suppose again that $t_1 = 8$ and also that credit constraints on the consumer, of the type described in Question 1, are in place, with lenders stipulating that consumers cannot be in debt at the end of period one (i.e., the credit constraint again takes the form $a_1 \geq 0$). Will this credit constraint affect consumers' optimal decisions? Explain why or why not. Is this credit constraint welfare-enhancing, welfare-diminishing, or welfare-neutral?
e. Now with the credit constraint described above in place, consider again the tax cut of 1 unit in the first period, with no change in government purchases. (That is, t_1 falls from 8 units to 7 units.) What is the change in national savings in period one that arises due to the tax cut? Provide economic intuition for the result you obtain.

Solution:

a. Using the government's LBC, find that $t_2 = 2.2$.
b. Using the same procedure as in question 1a above (specifically, starting with the condition that $u_1(c_1, c_2) = (1+r)u_2(c_1, c_2)$) and with the given functions, we get that at the optimal choice, $c_2 = (1+r)c_1$. Plugging this into the LBC of the economy and solving for c_1 yields $c_1 = \frac{1}{2}\left[y_1 - g_1 + \frac{y_2 - g_2}{1+r}\right]$, from which it immediately follows that $c_1 = 10$, which then implies that $c_2 = 1.1$.
c. With government purchases unchanged, a change in the timing of lump-sum taxes leads to no change in consumption and hence no change in national savings. This is the Ricardian Equivalence proposition – consumers increase their private savings after the tax cut in anticipation of the tax increase that must occur in period two.
d. Examine the period-one budget constraint of the consumer: $c_1 + a_1 = y_1 - t_1$ (remember, the consumer has zero initial assets here). This expression, along with the value of $c_1 = 10$ you found in part b above can be used to determine that $a_1 = -9$. Thus, consumers optimally (i.e., under no credit constraints) want to be debtors at the end of period one. With the imposition of the credit constraints, consumers can no longer do so, and will choose $a_1 = 0$ because that is the closest they can get to their unrestricted choice while also satisfying the credit constraint. The period one budget constraint, with $a_1 = 0$, yields $c_1 = y_1 - t_1 = 1$. The credit constraint diminishes welfare because consumers are being forced to choose a consumption allocation different from the one they would otherwise choose – graphically, they are on a lower indifference curve than the one that maximizes utility subject to the LBC of the economy.
e. With $t_1 = 7$, the credit constraint is still binding, and $c_1 = y_1 - t_1 = 2$. Thus, because $s_1^{nat} = y_1 - c_1 - g_1$, national savings falls by exactly the amount by which consumption rises, which is one. This occurs because Ricardian Equivalence fails if capital controls/borrowing constraints are binding (another reason, beyond distortionary taxes as described in the Lecture Notes, why Ricardian Equivalence fails). The reason here is that consumers were not at their unrestricted optimal choice to begin with – they wanted to consume more in period one than they were restricted to. Thus, any relaxation of their period one budget constraint (ie, in the form of lower taxes in period one) induces them to increase their consumption, dragging down national savings.

3. "Marginal Propensity to Consume" for Various Utility Functions. An old (Keynesian) idea in macroeconomics is the "marginal propensity to consume," abbreviated MPC. Briefly, the MPC is the fraction of current-period income that is consumed in the current period. For example, if income in period one is 2 and consumption in period one is 1.8, the $MPC = 0.9$.

Consider the standard two-period consumption-savings model in which the representative consumer has no control over his real labor income in periods 1 and 2, denoted by y_1 and

y_2, respectively. As usual, denote by r the real interest rate between period 1 and period 2, and assume the individual begins his life with zero initial assets ($A_0 = 0$). Make the additional assumption that in present-discounted-value terms, his real income in each of the two periods is the same – that is, $y_1 = \frac{y_2}{1+r}$. Using the LBC and the intertemporal optimality condition, derive for each of the following utility functions the "period-1 MPC" – that is, derive what fraction of period-1 real income the consumer devotes to period-1 consumption. (In other words, derive the coefficient MPC in the expression $c_1 = MPC \cdot y_1 + const$.) Note that not all of these utility functions satisfy the property that utility is strictly concave in both its arguments – but this is irrelevant for the exercise here. **(Hint:** Set up the Lagrangian in order to solve.)

a. $u(c_1, c_2) = \sqrt{c_1} + \sqrt{c_2}$, with $u_1 = \frac{1}{2\sqrt{c_1}}$ and $u_2 = \frac{1}{2\sqrt{c_2}}$

b. $u(c_1, c_2) = \ln(c_1) + c_2$ (No, this is not a typo…), with $u_1 = \frac{1}{c_1}$ and $u_2 = 1$

c. $u(c_1, c_2) = c_1^b c_2^{1-b}$**,** with $u_1 = bc_1^{b-1}c_2^{1-b}$ and $u_2 = (1-b)c_1^b c_2^{-b}$, where b is a constant such that $0 < b < 1$. (This type of utility function is called the Cobb-Douglas utility function).

General Comment: In all of the following, the important point that comes out of the solution of the Lagrangean is that at the optimal choice, the intertemporal MRS (ie, the ratio u_1 / u_2) equals $1+r$, a derivation that by now you should be familiar with (indeed, by now you should be familiar with at least this result without having to set up and solve the Lagrangean). Of course, you need to compute the marginal utility functions – above, the marginal utility functions for each given utility function are presented, which you needed to compute yourself.

So we have the consumption-savings optimality condition $\frac{u_1}{u_2} = 1+r$ along with the LBC here, which with our assumption of $y_1 = \frac{y_2}{1+r}$, is $c_1 + \frac{c_2}{1+r} = 2y_1$. The general solution procedure here is use the given functional forms with the optimality condition and the LBC to generate the MPC. Solving the LBC for c_1, $c_1 = 2y_1 - \frac{c_2}{1+r}$. We can then use the optimality condition to solve for c_2 in terms of c_1, substitute into the LBC, and generate the appropriate relation between c_1 and y_1.

a. $u(c_1, c_2) = \sqrt{c_1} + \sqrt{c_2}$, with $u_1 = \frac{1}{2\sqrt{c_1}}$ and $u_2 = \frac{1}{2\sqrt{c_2}}$

Solution: The optimality condition here states that $\frac{\sqrt{c_2}}{\sqrt{c_1}} = 1 + r$, from which we get that $c_2 = (1+r)^2 c_1$. Inserting this into the LBC and solving for c_1 we get

$$c_1 = \left(\frac{2}{2+r}\right) y_1.$$

Thus, $MPC = \frac{2}{2+r}$.

b. $u(c_1, c_2) = \ln(c_1) + c_2$ (No, this is not a typo…), with $u_1 = \frac{1}{c_1}$ and $u_2 = 1$

Solution: The optimality condition here states that $\frac{1}{c_1} = 1 + r$, which obviously is independent of c_2. Optimal period-one consumption is thus obviously $c_1 = \frac{1}{1+r}$, independent of y_1. Hence the period-one MPC for this utility function is zero (i.e., period-one consumption does not depend on period-one income – period-one consumption is said to be **autonomous** here.

c. $u(c_1, c_2) = c_1^{\ b} c_2^{\ 1-b}$**,** with $u_1 = bc_1^{b-1} c_2^{1-b}$ and $u_2 = (1-b)c_1^b c_2^{-b}$, where b is a constant such that $0 < b < 1$. (This type of utility function is called the Cobb-Douglas utility function).

Solution: The optimality condition here states that $\frac{b}{1-b}\frac{c_2}{c_1} = 1 + r$, from which we get that $c_2 = \frac{1-b}{b}(1+r)c_1$. Inserting this into the LBC and solving for c_1 we get

$$c_1 = 2 \cdot b \cdot y_1.$$

Thus, $MPC = 2b$.