Economics 602
Macroeconomic Theory and Policy
Problem Set 5 Suggested Solutions
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1. **Infrequent Stock Transactions.** Consider a representative consumer at time t seeking to maximize the sum of discounted lifetime utility from t on,

$$\sum_{s=0}^{\infty} \beta^s u(c_{t+s})$$

subject to the infinite sequence of flow budget constraints

$$P_t c_t + S_t a_t = S_t a_{t-2} + D_t a_{t-2} + Y_t,$$

where the notation is as in class: a_t is holdings of a real asset (a "stock") at the end of period t, S_t is its nominal price in t, D_t is the nominal dividend that each unit of assets carried into t **from period t-2** pays out, Y_t is nominal income in t, c_t is consumption in t, and P_t is the nominal price of each unit of consumption in t. Note well how the budget constraint is written: it is assets accumulated in period t-2 that pay off in period t – thus, in this model, stocks (for some reason…) must be held for two periods, rather than being able to be traded every period. Construct the Lagrangian to compute the stock price S_t in period t. Explain intuitively how and why the stock price differs from that in the model studied in class, in which all shares can be traded every period.

Solution: In differentiating the lifetime Lagrangian, we now need to look forward two periods from period t in order to see the consequences (payoffs) of period-t asset holding decisions. Specifically, the terms in the Lagrangian (between period t and t+2 inclusive) that involve period-t choice variables (namely, c_t and a_t) are:

$$\begin{aligned}
&\left[u(c_t) + \lambda_t (S_t a_{t-2} + D_t a_{t-2} + Y_t - P_t c_t - S_t a_t)\right] + \\
&\left[\beta u(c_{t+1}) + \beta \lambda_{t+1} (S_{t+1} a_{t-1} + D_{t+1} a_{t-1} + Y_{t+1} - P_{t+1} c_{t+1} - S_{t+1} a_{t+1})\right] + \\
&\left[\beta^2 u(c_{t+2}) + \beta^2 \lambda_{t+2} (S_{t+2} a_t + D_{t+2} a_t + Y_{t+2} - P_{t+2} c_{t+2} - S_{t+2} a_{t+2})\right] + \ldots
\end{aligned}$$

Note carefully these terms. The ellipsis at the end indicate that the summation continues forever (since the consumer is assumed to maximize **lifetime** utility), but the terms written down are only ones that are important for the problem at hand: the consumer in period t chooses c_t and a_t and there are no other terms in the Lagrangian (i.e., there are no other budget constraints) that contain these quantities. Also note the as we move successive periods into the future, the discount factor β is exponentiated further.

The first-order condition with respect to c_t is

$$u'(c_t) - \lambda_t P_t = 0,$$

which is standard. Nonstandard is the first-order condition with respect to a_t:

$$-\lambda_t S_t + \beta^2 \lambda_{t+2}(S_{t+2} + D_{t+2}) = 0.$$

The above two conditions (including the analogous first-order condition with respect to consumption at time t + 2) can be combined to give

$$\frac{u'(c_t)}{u'(c_{t+2})} = \beta^2 \left[\frac{S_{t+2} + D_{t+2}}{S_t} \cdot \frac{P_t}{P_{t+2}} \right],$$

which is analogous to the condition we derived in the model in class with a one-period holding period for assets, except the two-period holding period here means that it is period t + 2 that is the relevant future period in which to evaluate the marginal utility of consumption, stock price, dividend, and nominal price of consumption. Instead, we can solve for the period-t stock price,

$$S_t = \beta^2 \left[\frac{u'(c_{t+2})}{u'(c_t)} \left(S_{t+2} + D_{t+2} \right) \frac{P_t}{P_{t+2}} \right],$$

which indicates that it is period t + 2 marginal utility, price level, stock price, and dividend that affects the stock price in period t. This should make sense because (by assumption) stock purchased in period t does not yield anything until period t + 2, so the relevant decision horizon is a two-period horizon, which is reflected in the stock price in period t.

2. **House Prices.** With all the talk in the news the past few years of soaring and then crashing house prices, let's see how our simple multi-period model can be used to think of how house prices are determined. Suppose the instantaneous utility function is $u(c_t, h_t)$, where c_t as usual stands for consumption in period t, and now h_t stands for the level of housing services an individual enjoys in period t (i.e., the "quantity" of house an individual owns). Denote by H_t the nominal price of a house in period t. The quantity of house owned at the beginning of period t is h_{t-1}, and the quantity of house owned at the end of period t is h_t, and assume that the quantity of house can be changed every period (think of this loosely as making additions, repairs, etc to your house on a regular basis). Thus, we can write the flow budget constraint in period t as $P_t c_t + H_t h_t = H_t h_{t-1} + Y_t$, where Y_t is nominal income over which the consumer has no control. Note for simplicity we have omitted other assets from the model, houses are the only assets in this model. Solve for the nominal price of a house in period t, H_t. Discuss qualitatively why the marginal rate of substitution between housing services and consumption appears in the pricing equation. How is the setup of this asset-pricing model different from the setup of our "stock-pricing" model in class? How is it the same?

Solution: Setting up the Lagrangian for the maximization of utility from period t onwards in the usual way, the only terms that include c_t and h_t (which are the only objects of choice in period t) are

$$\left[u(c_t,h_t)+\lambda_t\left(H_t h_{t-1}+Y_t-P_t c_t-H_t h_t\right)\right]+$$
$$\left[\beta u(c_{t+1},h_{t+1})+\beta\lambda_{t+1}\left(H_{t+1}h_t+Y_{t+1}-P_{t+1}c_{t+1}-H_{t+1}h_{t+1}\right)\right]$$

The first-order condition with respect to h_t is

$$u_h(c_t,h_t)-\lambda_t H_t+\beta\lambda_{t+1}H_{t+1}=0,$$

where $u_h(\cdot)$ denotes the marginal utility function with respect to housing. Solving for the nominal house price H_t,

$$H_t=\frac{u_h(c_t,h_t)}{\lambda_t}+\frac{\beta\lambda_{t+1}H_{t+1}}{\lambda_t},$$

which shows us that H_t depends on a forward-looking term which involves, ass usual, the pricing kernel $\beta\lambda_{t+1}/\lambda_t$ and the future house price, but also a term that involves the marginal utility of housing. Returning to the Lagrangian, the first-order condition with respect to c_t is

$$u_c(c_t,h_t)-\lambda_t P_t=0,$$

from which we get that $\lambda_t=u_c(c_t,h_t)/P_t$ (note that this solution for the multiplier is the one we've most-often encountered, i.e., in our "standard" (no habit persistence and one-period asset-holding periods) sequential formulation problem. Inserting this value of λ_t (as well as its period-(t+1) counterpart) into the expression for the nominal house price, we have

$$H_t=\frac{u_h(c_t,h_t)P_t}{u_c(c_t,h_t)}+\frac{\beta u_c(c_{t+1},h_{t+1})}{u_c(c_t,h_t)}\cdot\frac{P_t}{P_{t+1}}\cdot H_{t+1}.$$

The term u_h/u_c is simply the marginal rate of substitution between housing services and consumption – i.e., it's simply the ratio of derivatives of the utility function with respect to its two arguments (and hence, graphically, it's the slope of the indifference curve defined in $c-h$ space….this should all at this point be review of the basics of consumer theory….).

The reason the MRS between housing and consumption appears in the pricing equation is that housing (the only asset in this model) appears directly in the utility function. The value (reflected in the price) of housing thus is affected by how much consumers prefer housing relative to consumption. Making an analogy with our Chapter 1 model of consumer theory, u_h/u_c measures how much consumption the individual is willing to give up in order to obtain one more unit of housing (think graphically here: recall that if we plotted c_1 on the horizontal axis and c_2 on the vertical axis, then the slope of the indifference curve, given by u_{c_2}/u_{c_1}, measured how much c_2 the individual was willing to give up to obtain one more unit of c_1). The higher is u_h/u_c, the more consumption the individual is willing to trade for housing (i.e., the more **desirable** housing is), the higher will its price be. (Then we need to scale this MRS by the nominal price level P_t because H_t is measured in nominal terms).

The term u_h / u_c can be thought of as the housing dividend. Recall from our basic stock-pricing model that stock paid out some dividend, which affected the price of the stock. In this case, the dividend is essentially the direct utility the house yields, and that direct utility can be thought of as the dividend. Thus, the fact that there is some dividend as in the basic stock-pricing model makes the model of house prices similar to our baseline model – the fact that the dividend is directly utility-based makes it different.

3. **Habit Persistence in Consumption.** An increasingly common utility function used in macroeconomic applications is one in which period-t utility depends not only on period-t consumption but also on consumption in periods earlier than period t. This idea is known as "habit persistence," which is meant to indicate that consumers become "habituated" to previous levels of consumption. To simplify things, let's suppose only period-(t-1) consumption enters the period-t utility function. Thus, we can write the instantaneous utility function as $u(c_t, c_{t-1})$. When a consumer arrives in period t, c_{t-1} of course cannot be changed (because it happened in the past).
   a. In a model in which stocks (modeled in the way we introduced them in class) can be traded every period, how is the pricing equation for S_t (the nominal stock price) altered due to the assumption of habit persistence? Consumption in which periods affects the period-t stock price under habit persistence? To answer this, derive the pricing equation using a Lagrangian and compare its properties to the standard model's pricing equation developed in class. Without habit persistence (i.e., our baseline model in class), consumption in which periods affects the stock price in period t?

Solution: Using the same notation developed in class and problem 2 of Problem Set 5: as usual, the consumer's choice variables in period t are c_t and a_t. The relevant terms in the consumer's Lagrangian from period t onwards are:

$$\left[u(c_t, c_{t-1}) + \lambda_t (S_t a_{t-1} + D_t a_{t-1} + Y_t - P_t c_t - S_t a_t)\right] +$$
$$\left[\beta u(c_{t+1}, c_t) + \beta\lambda_{t+1}(S_{t+1} a_t + D_{t+1} a_t + Y_{t+1} - P_{t+1} c_{t+1} - S_{t+1} a_{t+1})\right] + \ldots$$

Notice carefully the terms here: utility in period t depends on consumption in t and t-1; hence utility in period t+1 depends on consumption in t+1 and t; and the timing on the asset's returns are as in class, reflecting the one-period holding period.

The first-order condition with respect to a_t is simply

$$-\lambda_t S_t + \beta\lambda_{t+1}(S_{t+1} + D_{t+1}) = 0,$$

which can be rearranged as usual to give $S_t = \beta\left[\frac{\lambda_{t+1}}{\lambda_t}\left(S_{t+1}+D_{t+1}\right)\right]$. Thus far nothing is different from our baseline model in class. However, the way in which the Lagrange multiplier λ_t evolves over time is now more complicated. Take the first-order condition of the Lagrangian with respect to c_t to get

$$u_1(c_t, c_{t-1}) - \lambda_t P_t + \beta u_2(c_{t+1}, c_t) = 0,$$

where the notation u_i denotes the partial derivative of the instantaneous utility function with respect to the i-th argument (since the instantaneous utility function here has two arguments). Thus $u_1(c_t, c_{t-1})$ is the marginal utility **in period t of period t consumption,** while $u_2(c_{t+1}, c_t)$ is the marginal utility **in period t+1 of period t consumption** – that is, due to the habit persistence, period-t consumption affects utility in both periods t and t+1 (reflecting the "habit formation"), which must be taken into account. Solving the above for the Lagrange multiplier, we have $\lambda_t = \frac{u_1(c_t, c_{t-1}) + \beta u_2(c_{t+1}, c_t)}{P_t}$, and thus, updating the time subscripts by one period, similarly in period t+1, $\lambda_{t+1} = \frac{u_1(c_{t+1}, c_t) + \beta u_2(c_{t+2}, c_{t+1})}{P_{t+1}}$.

Inserting these two into the stock price equation above,

$$S_t = \beta\left[\frac{u_1(c_{t+1}, c_t) + \beta u_2(c_{t+2}, c_{t+1})}{u_1(c_t, c_{t-1}) + \beta u_2(c_{t+1}, c_t)} \cdot \frac{P_t}{P_{t+1}}\left(S_{t+1}+D_{t+1}\right)\right].$$

The period-t stock price is affected by not only c_{t+1} and c_t (the standard model in class), but also, due to habit persistence, c_{t+2} and c_{t-1}. Thus, with habit persistence, asset prices are said to be more **forward-looking** as well as more **backward-looking** than without habits, and this idea seems to better capture empirically the behavior of stock prices than the model without habit persistence (a topic for a more advanced course in finance theory).

b. Based on your solution in part a and the pattern you notice there, if the instantaneous utility function were $u(c_t, c_{t-1}, c_{t-2})$ (that is, two lags of consumption appear, meaning that period t utility depends on consumption in periods t, t-1, and t-2), consumption in which periods would affect the period-t stock price? No need to derive the result very formally here, just draw an analogy with what you found above.

Solution: With no habits (our baseline model in class), the price S_t depended on period t and period t+1 consumption. With one lag of consumption as the habit model (part a above), the price S_t depended on period t-1, t, t+1, and t+2 consumption. Thus, having one lag of consumption introduced one more backward-looking and one more forward-looking consumption term in the pricing equation. Adding yet one more lag to the habit model would introduce yet another backward-looking and yet another forward-looking consumption term in the pricing equation: thus, consumption in periods t-2, t-1, t, t+1, t+2, and t+3 would all affect the period-t stock price. And so on for even further lags of consumption in the period-t instantaneous utility function.