Economics 602
Macroeconomic Theory and Policy
Problem Set 6 Suggested Solutions
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Spring 2012

1. **Lags in Labor Hiring.** Rather than supposing that the representative firm at the beginning of period t can decide how much labor it would like to hire for use in period t, suppose that labor used in period t must be chosen in period t-1. (That is, suppose n is a stock (aka state) variable.) As usual, capital for use in production in period t must be purchased in period t-1 because of the "time to build" surrounding capital goods. With this lag in labor hiring, construct the lifetime (in the two-period model) profit function of the firm, and show that the real interest rate now is a relevant price for labor as well as capital goods. Provide brief economic intuition. (**Hint:** Make as close an analogy with our model of firm ownership of capital as you can – in particular, think of workers in this model as being "owned" (contractually obligated to) firms.)

Solution:

With employees being contractually bound to ("owned by") firms, the period-t nominal profits of a firm are given by

$$PR_t = P_t f(k_t, n_t) + P_t k_t + P_t w_t n_t - P_t k_{t+1} - P_t w_t n_{t+1},$$

in which labor used in production in period t, n_t, is chosen in period t-1 (and thus labor used in production in period t+1, n_{t+1}, is chosen in period t. In analogy with our model with only capital pre-determined, the employees of a firm are a valuable "asset," with total market value $P_t w_t n_t$ -- notice that this term enters **positively** in period t profits, rather than negatively with non-pre-determined labor. What enters negatively in period t profits here is the "purchase" of period t+1 labor, namely the term $-P_t w_t n_{t+1}$. In the two period model, discounted nominal profits of the firm are therefore

$$PR = P_1 f(k_1, n_1) + P_1 k_1 + P_1 w_1 n_1 - P_1 k_2 - P_1 w_1 n_2 + \frac{P_2 f(k_2, n_2)}{1+i_i} + \frac{P_2 k_2}{1+i_i} + \frac{P_2 w_2 n_2}{1+i_i} - \frac{P_2 k_3}{1+i_i} - \frac{P_2 w_2 n_3}{1+i_i}$$

The usual zero-terminal-assets condition in this case means that $k_3 = 0$ **and** $n_3 = 0$ (the latter, again, because labor should be thought of as an "asset" here). Focusing attention on the choice of n_2 (since n_1 was chosen in period t-1), the first-order condition of the lifetime profit function with respect to n_2 is

$$-P_1 w_1 + \frac{P_2 f_n(k_2,n_2)}{1+i_1} + \frac{P_2 w_2}{1+i_1} = 0 .$$

This expression can be rearranged to yield (using the exact Fisher equation)

$$(1+r_1)w_1 = f_n(k_2,n_2) + w_2 .$$

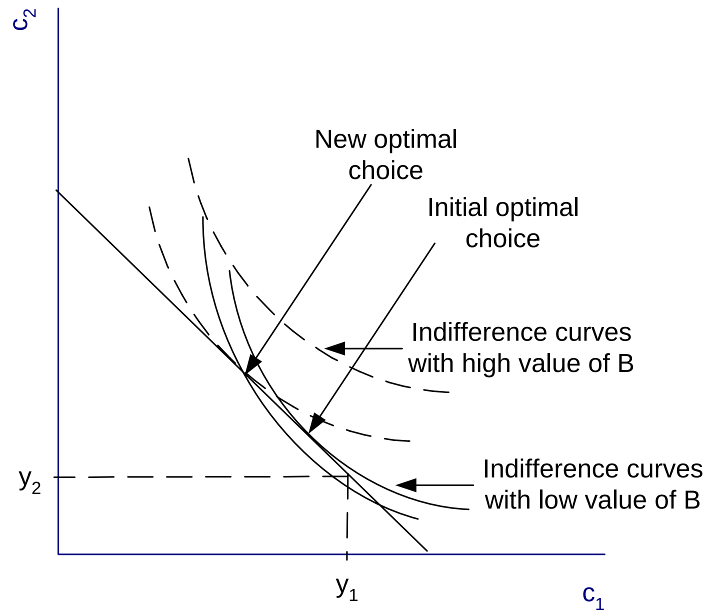
If the real wage were equal to one in each period, this condition would reduce to $r_1 = f_n(k_2,n_2)$, which would be almost identical to the condition we derived in class regarding capital demand (except of course in that case f_k is the relevant marginal product rather than f_n). The expression $r_1 = f_n(k_2,n_2)$ shows that if firms must choose labor for period 2 in period 1, the real interest rate between period 1 and period 2 is a relevant price to consider – which makes sense because there is now an interest opportunity cost associated with hiring labor (ie, "investment" in hiring).

However, in general of course w_1 and w_2 are not one, hence the above condition is not exactly the same as the capital demand condition. In the capital demand condition, the real price of capital goods is the same as the real price of consumption (which is one…) – note the discussion on p. 70-71 of the Lecture Notes describing that because capital goods and consumption goods are assumed to be the same goods (ie, computers can be viewed as both consumption goods and capital goods), the dollar price of each in our theoretical model is the same. The same is not true of labor – the nominal price of labor is W, which in general is different from P.

2. **Preference Shocks in the Consumption-Savings Model.** In the two-period consumption-savings model (in which the representative consumer has no control over his real labor income y_1 and y_2), suppose the representative consumer's utility function is $u(c_1, Bc_2)$, where, as usual, c_1 denotes consumption in period 1, c_2 denotes consumption in period 2, and B is a preference parameter.

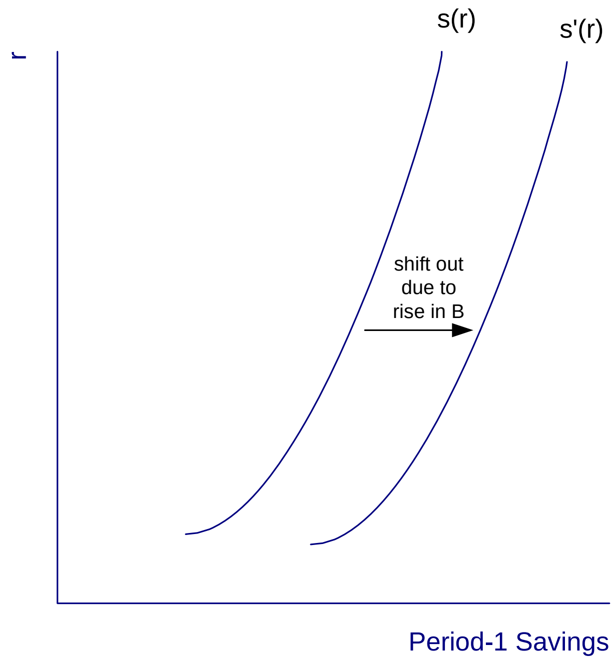
    a. Use an indifference-curve/budget-constraint diagram to illustrate the effect of an increase in B on the consumer's optimal choice of period-1 consumption.

Solution: An increase in B means each unit of period-2 consumption delivers more utility to the consumer. Thus, in utility terms, period-2 consumption has now become more valuable relative to period-1 consumption, implying that in order to stay on a given indifference curve the consumer now needs to give up fewer units of c_2 in order to get one more unit of c_1. In a diagram with c_2 on the vertical axis and c_1 on the horizontal axis, this is represented by a flattening of the indifference map. Because the LBC is unaffected, the flattening of the indifference map means that the new optimal choice features smaller period-1 consumption and hence larger period-2 consumption, as shown in the accompanying diagram. As drawn, consumption in period 1 is smaller than real income in period 1, but that is irrelevant.



b. Illustrate the effect of an increase in B on the private savings function. Provide economic interpretation for the result you find.

Solution: We can deduce the effect on private savings in period 1 using the diagram in part a above. The real interest rate has not changed (in other words, the slope of the LBC has not changed), yet the representative consumer's savings in period 1 has increased. This follows directly from the observation that income y_1 is constant while consumption in period 1 falls. This result would be true for any choice of the real interest rate (in other words, no matter the slope of the LBC), hence the private savings function shifts outwards, as shown below.



c. In the months preceding the U.S. invasion of Iraq, data shows that consumers decreased their consumption and increased their savings. Is an increase in B and the effects you analyzed in parts a and b above consistent with the idea that consumption fell and savings increased because of a looming war? If so, explain why; if not, explain why not.

Solution: Yes, these effects are consistent with developments in consumption and savings behavior in the U.S. leading up to the invasion of Iraq. An interpretation we can give using the model here is that consumers believed future macroeconomic conditions would be better than current (i.e., just before the war) macroeconomic conditions, hence a fall in consumption in the present (period 1) accompanied by a (expected) rise in consumption in the future (period 2). With B pre-multiplying consumption in the utility function (in the case here, period-2 consumption), the term B can be interpreted as a measure of "consumer confidence": a rise in B signals that consumers are shifting their preferences towards consumption (in that period). So here, we might interpret events as consumers being more confident about the future than the present, hence they postpone some consumption until the future..

d. Using a Lagrangian and assuming the utility function is $u(c_1, B \cdot c_2) = \ln(c_1) + \ln(B \cdot c_2)$, show how the representative consumer's MRS (and hence optimal choices of consumption over time) depends on B.

Solution: Setting up the Lagrangian in the two-period model as always, we have

$$\ln(c_1) + \ln(B \cdot c_2) + \lambda \left[y_1 + \frac{y_2}{1+r_1} - c_1 - \frac{c_2}{1+r_1} \right],$$

in which for simplicity we have assumed the initial assets equal zero because it does not at all affect the consumption-savings optimality condition (verify this yourself). The FOCs on c_1 and c_2 are, respectively,

$$\frac{1}{c_1} - \lambda = 0$$

$$\frac{B}{B \cdot c_2} - \frac{\lambda}{1+r_1} = 0$$

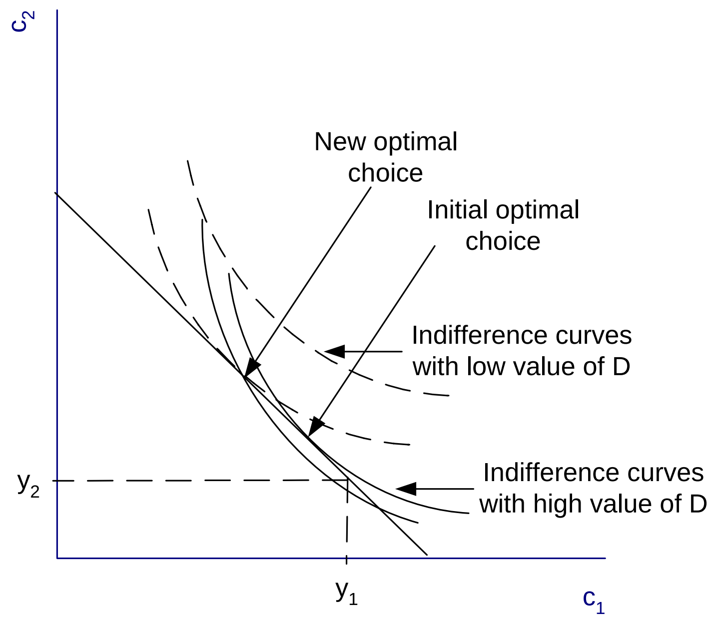
In the FOC on c_2, note that the B term ends up canceling out (because, recall, the derivative of an expression such as $\ln(2x)$ is $2/(2x) = 1/x$). Combining these two FOCs as usual then yields that at the optimal choice,

$$\frac{1/c_1}{1/c_2} = 1 + r_1,$$

the left-hand-side of which is the intertemporal MRS, as always. Note that it is **independent** of the preference shifter B, which turns out to be a special feature of the log utility function.

e. How would your analysis in parts a and b change if the consumer's utility function were $u(Dc_1, c_2)$ (instead of $u(c_1, Bc_2)$) and you were told that the value D decreased? (D is simply some other measure of preference shocks.)

Solution: Here, we return to a general utility specification, not necessarily log. With the utility function written as $u(Dc_1, c_2)$ and a decrease in D, the analysis above is completely unchanged. The fall in D makes consumption in period 1 less valuable in utility terms relative to period-2 consumption, which means that in order to obtain one more unit of period-2 consumption while remaining on the same indifference curve the consumer must give up more units of period-1 consumption than he had to before the fall in D. But in a diagram with c_2 on the vertical axis and c_1 on the horizontal axis, this simply means that the indifference curves become flatter, just as in part a.



This exercise cautions you to think about the underlying economics – specifically, how the consumer's marginal rate of substitution (refer to Chapter 1) is affected – when analyzing preference shocks. We cannot make a blanket statement such as "the indifference map flattens when the measure of the preference shock increases" because it depends on exactly how we introduce the preference shock into our theoretical model. Here in part d we introduced the preference shock by attaching it to period-1 consumption, whereas earlier we introduced the preference shock by attaching it to period-2 consumption.

3. **Intertemporal Consumption-Leisure Model – A Numerical Look.** Consider the intertemporal consumption-savings model. Suppose the lifetime utility function is given by $v(B_1c_1, l_1, B_2c_2, l_2) = u(B_1c_1, l_1) + u(B_2c_2, l_2)$, which is a slight modification of the utility function presented in Chapter 5. The modification is that preference shifters B_1 and B_2 enter the lifetime utility function, with B_1 the preference shifter in period one and B_2 the preference shifter in period two. In each of the two periods the function u takes the form

$$u(B_tc_t, l_t) = 2\sqrt{B_tc_t} + 2\sqrt{l_t}\ .$$

Note the t subscripts -- $t = 1,2$ depending on which period we are considering. Labor tax rates, real wages, the real interest rate between period one and period two, and the preference realizations are given by: $t_1 = 0.15$, $t_2 = 0.2$, $w_1 = 0.2$, $w_2 = 0.25$, $r = 0.15$, $B_1 = 1$, $B_2 = 1.2$. Finally, the initial assets of the consumer are zero.

Solution: Note that you needed to compute the marginal utility functions. For the given lifetime utility function, the marginal utility functions are, for $t = 1,2$:

$$v_{c_t} = \frac{\sqrt{B_t}}{\sqrt{c_t}};\ v_{l_t} = \frac{1}{\sqrt{l_t}}$$

a. Construct the marginal rate of substitution functions between consumption and leisure in each of period one and period two (**Hint:** these expressions will be functions of consumption and leisure – you are not being asked to solve for any numerical values yet). How does the preference shifter affect this intratemporal margin?

Solution: As by now is routine, the consumption-leisure marginal rate of substitution function is $MRS_{c_t l_t} = v_{l_t} / v_{c_t}$. With the given functions, the marginal rate of substitution function in period t, where t is either 1 or 2, is thus

$$MRS_{c_t l_t}(c_t, l_t) = \frac{\sqrt{c_t}}{\sqrt{B_t}\sqrt{l_t}}.$$

Again, note that this function is the MRS function for period $t = 1, 2$. From this function it is clear that a rise in B_t lowers this MRS, meaning a rise in B_t flattens the indifference map over consumption and leisure within a given period.

b. Construct the marginal rate of substitution function between period-one consumption and period-two consumption. (**Hint:** Again, you are not being asked to solve for any numerical values yet.) How do the preference shifters affect this intertemporal margin?

Solution: Again as by now should be routine, the intertemporal MRS function is given by $MRS_{c_1 c_2} = v_{c_1} / v_{c_2}$. Note the subscripts: v_{c_1} denotes the marginal utility function with respect to period-one consumption, and v_{c_2} denotes the marginal utility function with respect to period-two consumption. Using the given v_c function, we have

$$MRS_{c_1 c_c}(c_1, c_2) = \frac{\sqrt{B_1}}{\sqrt{B_2}} \cdot \frac{\sqrt{c_2}}{\sqrt{c_1}}.$$

The ratio of B values across the two periods affects the slope of the indifference map between period-one and period-two consumption. The larger is the ratio B_1 / B_2, the steeper is the indifference map across consumption in the two periods – the interpretation of this is that the larger is B_1 relative to B_2, the more "confident" (recall our interpretation of B from class) consumers are about the present (period one) than they are about the future (period two), hence the more period-two consumption they are willing to give up for a given increase in period-one consumption (which is our usual interpretation of the slope of an indifference curve with c_1 plotted on the horizontal axis and c_2 plotted on the vertical axis).

c. Using the expressions you developed in parts a and b along with the lifetime budget constraint (expressed in **real** terms…) and the given numerical values, solve numerically for the optimal choices of consumption in each of the two periods and of leisure in the two periods. (**Hint:** You need to set up and solve the appropriate Lagrangian.) (**Note:** the computations here are messy and the final answers do not necessarily work out "nicely." **To preserve some numerical accuracy, carry out your computations to at least four decimal places.**)

Solution: The LBC in real terms is

$$c_1 + \frac{c_2}{1+r} = (1-t_1)w_1(168 - l_1) + \frac{(1-t_2)w_2(168-l_2)}{1+r}. \tag{0.1}$$

This expression follows readily from expression (34) on p. 60 of the Lecture Notes (it's probably a good idea to derive this from expression (34) if you don't see it immediately), with zero initial assets imposed. This LBC involves the four unknowns, c_1, c_2, l_1, and l_2, which are the variables you are asked to solve for. We need three other expressions involving these variables – these three are the two consumption-leisure optimality conditions (one for each of period one and period two) and the one consumption-savings optimality condition. **By now you should know how these optimality conditions can be obtained by formulating the appropriate Lagrangian –** for ease of exposition the Lagrangian is omitted here. Suffice it to say it is simply the above consumption-leisure and consumption-savings optimality conditions that emerge from the Lagrangian. The consumption-leisure optimality conditions for period one and period two and the consumption-savings optimality condition are, respectively,

$$MRS_{c_1 l_1} = \frac{\sqrt{c_1}}{\sqrt{B_1}\sqrt{l_1}} = (1-t_1)w_1, \tag{0.2}$$

$$MRS_{c_2 l_2} = \frac{\sqrt{c_2}}{\sqrt{B_2}\sqrt{l_2}} = (1-t_2)w_2, \tag{0.3}$$

$$MRS_{c_1 c_2} = \frac{\sqrt{B_1}}{\sqrt{B_2}}\frac{\sqrt{c_2}}{\sqrt{c_1}} = 1+r. \tag{0.4}$$

By now you should know the interpretation of these optimality conditions: they simply represent the tangency between a relevant budget constraint and a relevant indifference curve. Equations (0.1), (0.2), (0.3), and (0.4) are now four equations in the four unknowns c_1, c_2, l_1 and l_2, so we can solve with some algebraic effort.

Let's decide to express the unknowns c_2, l_1, and l_2 all in terms of c_1. Once we have done this, we can substitute into the LBC and solve for c_1. From (0.4), we get that

$$c_2 = \frac{B_2}{B_1}(1+r)^2 c_1; \tag{0.5}$$

from (0.2), we get that

$$l_1 = \left(\frac{1}{(1-t_1)w_1}\right)^2 \frac{1}{B_1} c_1; \tag{0.6}$$

and from (0.3) we similarly get that

$$l_2 = \left(\frac{1}{(1-t_2)w_2}\right)^2 \frac{1}{B_2} c_2. \tag{0.7}$$

In (0.7), we need to substitute out c_2 using (0.5) (because, recall, we are trying to express the unknowns in terms of c_1), giving us

$$l_2 = \left(\frac{1+r}{(1-t_2)w_2}\right)^2 \frac{1}{B_1} c_1. \tag{0.8}$$

Now, substitute into the LBC using (0.5), (0.6), and (0.8). Doing so and collecting all the resulting terms involving c_1 on the left-hand-side (you should perform these algebraic steps yourself….) gives us

$$c_1 \cdot \left[1 + \frac{B_2}{B_1}(1+r) + \frac{1}{(1-t_1)w_1 B_1} + \frac{1+r}{(1-t_2)w_2 B_1}\right] = 168(1-t_1)w_1 + \frac{168(1-t_2)w_2}{1+r}, \tag{0.9}$$

in which the only unknown, as desired, is c_1. Inserting all of the given numerical values, we finally find that $c_1^* = 4.1233$. Then using (0.5), (0.6), and (0.8) we find $c_2^* = 6.5437$, $l_1^* = 142.6754$, and $l_2^* = 136.3272$. The individual thus works 168 – 142.6754 = 25.3246 hours per week in the first period and 168 – 136.3272 = 31.6728 hours per week in the second period.

d. Based on your answer in part c, how much (in real terms) does the consumer save in period one? What is the asset position that the consumer begins period two with?

Solution: Recall that real private savings (inclusive of taxes is) income minus tax payments minus consumption. Given the solution above, total real income in period one is $(168 - l_1)w_1 = 5.0649$, of which the amount paid in taxes is $t_1(168 - l_1)w_1 = 0.7597$. Disposable income (gross income less taxes) in period one is thus 5.0649 – 0.7597 = 4.3052. Subtracting period-one consumption, we have that real savings in period one is 4.3052 – 4.1233 = 0.1819. Because the consumer began period one with zero assets, at the end of period one his real asset position is thus 0.1819. (Then, with positive assets to begin period two, the individual is able to consume more than his income in period two – perform this calculation to verify this for yourself.)

e. Suppose B_2 were instead higher, at 1.6. How are your solutions in parts c and d affected? Provide brief interpretation in terms of "consumer confidence."

Solution: Examining the solution (0.9), we see that B_2 enters the solution in only one place. It is easy to conclude from (0.9) that a higher value of B_2 will lead to a lower value of optimal period-one consumption. Specifically, $c_1^* = 3.9923$, which then implies $c_2^* = 8.4476$, $l_1^* = 138.1405$, and $l_2 = 131.9941$.

With B_2 higher relative to B_1 (and with the particular way B enters the utility function, specifically, multiplying c), the consumer is more confident about the economic state in the future (period two) than in the present (period one). He thus works and consumes less in period one, and works and consumes more in period two due to the rise in B_2. Savings in period one rises to $(1-t_2)w_2(168-l_2)-c_1 = 1.0839$, consistent with the increased desire to postpone consumption until period two.

4. **Search Theory and Labor Demand.** The 2010 Nobel Prize in Economics was awarded to Peter Diamond, Dale Mortensen, and Christopher Pissarides for their development (during the 1970s and 1980s) of search theory. Search theory is a framework especially suited for studying labor market issues. The search framework builds on, but is richer than, the basic theory of supply and demand. Search theory can be applied to both the supply side of the labor market (building on the analysis of Chapter 2) as well as the demand side of the labor market (building on the analysis of Chapter 6). In what follows, you will study the application of search theory to the demand side of the labor market.

There are three basic ideas underlying search theory. First, search theory incorporates into basic supply-and-demand analysis the fact that when a firm wants to hire a worker (i.e., "demands labor"), there is a chance that a suitable worker may not be found. That is, a firm "searching" for a worker has a **probability less than one** that a suitable "match" will be found.

Second, search theory makes explicit the **costs associated with search activity.** As is realistic, when a firm wants to hire a worker, it does not simply "go to the market" as in basic supply-and-demand analysis. Rather, it must expend resources **searching** for a worker (think of these costs as the recruiting costs inherent in running a firm's human resources department, placing job advertisements in various outlets, the interviewing process, etc.). Moreover, because of the various activities involved in the search, or "recruiting," process, there is a time delay between when a firm engages in recruiting activities and when, if recruiting is successful, a new employee actually begins working at the firm. For concreteness, suppose that if a firm successfully recruited a new employee in period t, it is not until period t+1 that the new employee actually begins working.

Third, due to recruiting costs and time delays in the recruiting process, if a firm successfully hires a new employee, that employee will (typically) work for the firm for more than just one period. That is, in labor markets, **multi-period relationships between workers and firms** are the norm rather than the exception. (For example, in the U.S., the average length of time a worker remains in a particular job is between two and three years.)

To formalize these three ideas in the context of an **infinite-period firm profit-maximization problem,** introduce some notation:

Problem 4 continued

p_t^{FIND}: the **probability**, in period *t*, that a firm searching to fill a particular job opening finds a suitable worker. By the definitions of probabilities, $p_t^{FIND} \in [0,1]$. (Define in an analogous way p_{t+1}^{FIND}, p_{t+2}^{FIND}, p_{t+3}^{FIND}, and so on.)

J_t: the "recruiting cost," in period *t*, measured in real terms, that is associated with **each** job opening that a firm is trying to fill; the recruiting cost is $J_t \geq 0$. (Define in an analogous way J_{t+1}, J_{t+2}, J_{t+3}, and so on.)

$p_t^{TURNOVER}$: the **probability** that a worker **employed in a particular job in period *t*** will **NOT** be employed at that same job in period *t*+1 (whether due to quitting or being fired, each of which is a form of worker "turnover"). By the definitions of probabilities, $p_t^{TURNOVER} \in [0,1]$. (Define in an analogous way $p_{t+1}^{TURNOVER}$, $p_{t+2}^{TURNOVER}$, $p_{t+3}^{TURNOVER}$, and so on.)

v_t: the number of job **vacancies** in period *t* that a firm is attempting to fill (that is, the number of job openings the firm has and is actively recruiting for). The cost of "setting up" each vacancy (the administrative cost associated with recruiting) is the cost J_t described above. (Define in an analogous way v_{t+1}, v_{t+2}, v_{t+3}, and so on.)

Supposing that the (representative) firm has "many" employees, the way in which the total number of employees that it has on its payroll changes from period *t* to period *t*+1 is

$$n_{t+1} = \left(1 - p_t^{TURNOVER}\right) n_t + v_t p_t^{FIND},$$

which is to be understood as follows: the number of employees that work at the firm in period *t* is n_t, and then, because some new workers are hired in period *t* and some existing workers turn over between period *t* and *t*+1, the firm has a (possibly different) number of employees in period *t*+1, n_{t+1}. Similarly, the way in which the total number of employees that the firm has on its payroll changes from period *t*+1 to period *t*+2 is

$$n_{t+2} = \left(1 - p_{t+1}^{TURNOVER}\right) n_{t+1} + v_{t+1} p_{t+1}^{FIND},$$

the way in which the total number of employees that the firm has on its payroll changes from period *t*+2 to period *t*+3 is

$$n_{t+3} = \left(1 - p_{t+2}^{TURNOVER}\right) n_{t+2} + v_t p_{t+2}^{FIND},$$

Problem 4 continued

and so on. Refer to these previous three expressions (along with their analogs in periods t+3, t+4, etc) as the firm's **employment constraints.**

To complete the description of the (representative) firm's (dynamic) profit-maximization problem starting from the beginning of period t:

- In period t, total output and hence total revenue of the firm (denominated in real terms) is $A_t f(k_t, n_t)$, in which A_t denotes total factor productivity (TFP) in period t, k_t is the amount of capital ("machines and equipment") the firm has at the start of period t (recall from Chapter 6 that capital is a stock variable that "takes one period to build"), and $f(k_t, n_t)$ is the firm's production function. Similarly, for period t+1, total output and hence total revenue of the firm (denominated in real terms) is $A_{t+1} f(k_{t+1}, n_{t+1})$, and so on.
- The real interest rate between any two consecutive time periods is **always r > 0** (that is, the real interest does not change over time, which is indicated by the lack of a time subscript).
- The real wage the firm must pay each worker in period t is w_t, which is taken as given by the firm. Similarly, the real wage the firm must pay each worker in period t+1 is w_{t+1}, in period t+2 is w_{t+2}, and so on.
- The variables **taken as given** by the firm are real wages, the probabilities of finding workers, and the probabilities of worker turnover. That is, the firm takes $\left(w_t, p_t^{FIND}, p_t^{TURNOVER}\right)$ as given in period t, takes $\left(w_{t+1}, p_{t+1}^{FIND}, p_{t+1}^{TURNOVER}\right)$ as given in period t+1, takes $\left(w_{t+2}, p_{t+2}^{FIND}, p_{t+2}^{TURNOVER}\right)$ as given in period t+2, and so on.
- In period t, the firm's profit function (in real terms) is

$$A_t f(k_t, n_t) + k_t - w_t n_t - k_{t+1} - J_t v_t,$$

which, except for the inclusion of "total recruiting costs" $J_t v_t$, is identical to the analysis in Chapter 6. Thus, the firm's profit function (in real terms) in period t+1 is $A_{t+1} f(k_{t+1}, n_{t+1}) + k_{t+1} - w_{t+1} n_{t+1} - k_{t+2} - J_{t+1} v_{t+1}$, and so on for periods t+2, t+3, etc.

With this background in place, your analysis is to proceed as follows.

Problem 4 continued

a. Construct an infinite-period Lagrangian (starting from the beginning of period *t*) for the representative firm's (infinite-period) profit-maximization problem. **This Lagrangian must take into account the employment constraints described above,** along with correctly incorporating all of the other pieces of the theory described above. Use λ_t as your notation for the Lagrange multiplier on the period-*t* employment constraint, λ_{t+1} as the Lagrange multiplier on the period-(*t*+1) employment constraint, and so on. **Because the Lagrangian has an infinite number of terms, write out the first several terms to make clear the nature of the Lagrangian, and provide any explanation needed in constructing the Lagrangian. (Note 1:** use the two-period analysis of firm theory in Chapter 6 as your intuitive basis for constructing the Lagrangian.) **(Note 2:** the Lagrangian is critical for all of the analysis that follows, so you should make sure that your work here is absolutely correct!)

Solution: The infinite sequence of employment constraints is the set of constraints on the profit maximization problem. The profit-maximization problem begins from the perspective of the start of period t, which means that profit terms **and employment constraints beyond period t must be appropriately discounted.** That is, everything about future periods must be appropriately discounted, just as in Chapter 8, in which it was **both** utility **and** budget constraints beyond period t that were discounted.

Recall that in Chapter 8, the appropriate discount factors, as we look successively down the timeline beyond period t, were β, β^2, β^3, β^4, … In the firm optimization problem here, something similar is required. However, it is not successive powers of β that are needed (since the firm does not solve a **utility** maximization problem), but rather successive powers of $\frac{1}{1+r}$ that are needed. This follows from making an analogy between our two-period firm analysis in Chapter 6 and our infinite-period consumer analysis in Chapter 8: in our two-period firm analysis, recall that, in order to convert period-2 profits into present-discounted period-1 one terms, the "discount factor" (for the firm) $\frac{1}{1+r}$ was used. This was a one-period discounting – i.e., discounting period-2 profits back one period to period 1. By analogy, discounting period-3 profits back two periods to period 1 would require the "discount factor" $\left(\frac{1}{1+r}\right)^2 = \frac{1}{(1+r)^2}$; discounting period-4 profits back three periods to period 1 would require the "discount factor" $\left(\frac{1}{1+r}\right)^3 = \frac{1}{(1+r)^3}$, and so on.

Then, let the **non-discount-factor** component of the multiplier on the period-t employment constraint be λ_t, the **non-discount-factor** component of the multiplier on the period-(t+1) employment constraint be λ_{t+1} , the **non-discount-factor** component of the multiplier on the period-(t+2) employment constraint be λ_{t+2} , and so on.

Putting all of the above logic together, we have the Lagrangian for the profit maximization problem

$$
\begin{aligned}
&A_t f(k_t, n_t) + k_t - w_t n_t - k_{t+1} - J_t v_t + \left(\frac{1}{1+r}\right)\left[A_{t+1} f(k_{t+1}, n_{t+1}) + k_{t+1} - w_{t+1} n_{t+1} - k_{t+2} - J_{t+1} v_{t+1}\right] \\
&+\left(\frac{1}{(1+r)^2}\right)\left[A_{t+2} f(k_{t+2}, n_{t+2}) + k_{t+2} - w_{t+2} n_{t+2} - k_{t+3} - J_{t+2} v_{t+2}\right] \\
&+\left(\frac{1}{(1+r)^3}\right)\left[A_{t+3} f(k_{t+3}, n_{t+3}) + k_{t+3} - w_{t+3} n_{t+3} - k_{t+4} - J_{t+3} v_{t+3}\right] + \text{...(infinite number of terms)} \\
&+\lambda_t\left[\left(1 - p_t^{TURNOVER}\right) n_t + v_t p_t^{FIND} - n_{t+1}\right] + \left(\frac{\lambda_{t+1}}{1+r}\right)\left[\left(1 - p_{t+1}^{TURNOVER}\right) n_{t+1} + v_{t+1} p_{t+1}^{FIND} - n_{t+2}\right] \\
&+\left(\frac{\lambda_{t+2}}{(1+r)^2}\right)\left[\left(1 - p_{t+2}^{TURNOVER}\right) n_{t+2} + v_{t+2} p_{t+2}^{FIND} - n_{t+3}\right] \\
&+\left(\frac{\lambda_{t+3}}{(1+r)^3}\right)\left[\left(1 - p_{t+3}^{TURNOVER}\right) n_{t+3} + v_{t+3} p_{t+3}^{FIND} - n_{t+4}\right] + \text{...(infinite number of terms)}
\end{aligned}
$$

b. Based on the Lagrangian in part a, compute the first-order condition with respect to k_{t+1} (that is, with respect to how much capital the firm would optimally like to use in its production process in period *t*+1).

Solution: Given the effort of constructing the Lagrangian, the first-order conditions are easy to compute. The FOC with respect to k_{t+1} is

$$-1 + \left(\frac{1}{1+r}\right)\left[A_{t+1} f_k(k_{t+1}, n_{t+1}) + 1\right] = 0 .$$

c. Based on the first-order condition computed in part b, explain (using any appropriate combination of mathematical analysis, graphical analysis, and logic) how the "search" aspects of labor markets affect the firm's capital demand decisions. **For this part of the problem, you may (but do not need to) suppose that the production function is Cobb-Douglas:** $f(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$, with $\alpha \in (0,1)$.

Solution: The FOC in part b is identical to the FOC on capital accumulation from our analysis of the baseline two-period firm profit-maximization problem. Thus, search aspects of labor markets do not at all alter the capital demand function (recall that the FOC in part b **is** the capital demand function, when solved for *r*).

d. Based on the Lagrangian in part a, compute the following three first-order conditions: with respect to v_t, with respect to n_{t+1}, and with respect to v_{t+1} (that is, with respect to how many job openings ("vacancies") the firm optimally chooses in period *t* and period *t*+1, and how many employees the firm would optimally like to have on its payroll at the beginning of period *t*+1).

Solution: Simply reading the Lagrangian in part a, the FOC with respect to v_t is

$$-J_t + \lambda_t p_t^{FIND} = 0;$$

the FOC with respect to n_{t+1} is

$$\left(\frac{1}{1+r}\right)\left(A_{t+1} f_n(k_{t+1}, n_{t+1}) - w_{t+1}\right) - \lambda_t + \left(\frac{\lambda_{t+1}}{1+r}\right)\left(1 - p_{t+1}^{TURONVER}\right) = 0;$$

and the FOC with respect to v_{t+1} is

$$-\left(\frac{1}{1+r}\right) J_{t+1} + \left(\frac{\lambda_{t+1}}{1+r}\right) p_{t+1}^{FIND} = 0.$$

e. Based on the three first-order conditions computed in part d, construct an expression that reads

$$\frac{J_t}{p_t^{FIND}} = \ldots,$$

in which the right-hand-side of the expression is for you to determine. **Your final expression may NOT include any Lagrange multipliers in it. (**You should make very clear the algebraic steps involved in constructing this expression.)

Solution: The first FOC in part d can be solved for the multiplier: $\lambda_t = \dfrac{J_t}{p_t^{FIND}}$.

Similarly, the third FOC in part d can be solved for the multiplier: $\lambda_{t+1} = \dfrac{J_{t+1}}{p_{t+1}^{FIND}}$, (Note the recursive property that emerges here, which, as we saw in Chapters 3, 4, and 8, is a property that emerges from any sequential Lagrangian analysis.)

Substituting these expressions for λ_t and λ_{t+1} into the second FOC in part d gives

$$\left(\frac{1}{1+r}\right)\left(A_{t+1}f_n(k_{t+1},n_{t+1})-w_{t+1}\right)-\frac{J_t}{p_t^{FIND}}+\left(\frac{J_{t+1}/p_{t+1}^{FIND}}{1+r}\right)\left(1-p_{t+1}^{TURONVER}\right)=0\,.$$

Getting to the final requested expression requires one further rearrangement. Isolate the term $\frac{J_t}{p_t^{FIND}}$, which gives

$$\frac{J_t}{p_t^{FIND}}=\left(\frac{1}{1+r}\right)\left(A_{t+1}f_n(k_{t+1},n_{t+1})-w_{t+1}\right)+\left(\frac{J_{t+1}/p_{t+1}^{FIND}}{1+r}\right)\left(1-p_{t+1}^{TURNOVER}\right).$$

It was fine if you stopped here, since this does satisfy the requested form. However, let's go one step further, and factor the term $\frac{1}{1+r}$ (which is just the discount factor) out of the right hand side, which gives

$$\frac{J_t}{p_t^{FIND}}=\left(\frac{1}{1+r}\right)\left(A_{t+1}f_n(k_{t+1},n_{t+1})-w_{t+1}+\left(1-p_{t+1}^{TURNOVER}\right)\frac{J_{t+1}}{p_{t+1}^{FIND}}\right).$$

In the rest of the analysis, let's refer to this as the **job creation condition.**

Problem 4 continued

The expression obtained in part e is known as the **job creation condition,** which is the central analytical prediction of search theory. In the remainder of the analysis, you will compare and contrast the job creation condition with the "labor demand" condition studied in Chapter 6. For the remainder of the analysis, **you may (but do not need to) suppose that the production function is Cobb-Douglas:** $f(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$, with $\alpha \in (0,1)$.

f. Consider the job creation condition in part e. Suppose that all workers turn over every period – that is, suppose

$$p_t^{TURNOVER} = p_{t+1}^{TURNOVER} = p_{t+2}^{TURNOVER} = p_{t+3}^{TURNOVER} = ...1.$$

With this assumption, what does the job creation condition simplify to? Briefly, but carefully, describe the economic interpretation of the job creation condition in this case?

Solution: Imposing $p_{t+1}^{TURNOVER} = 1$ on the job creation condition obtained in part e, we have

$$\frac{J_t}{p_t^{FIND}} = \left(\frac{1}{1+r}\right)\left(A_{t+1} f_n(k_{t+1}, n_{t+1}) - w_{t+1}\right).$$

To interpret this condition (or even the full job creation condition in part e), think in terms of basic "marginal benefit equals marginal cost" terms. This basic principle is the basis for economic interpretation/intuition of the result of **any** economic optimization analysis (not just with respect to firm theory or even just macroeconomic analysis).

The right hand side of the condition immediately above is the **present discounted value of the marginal benefit to a firm of successfully recruiting a worker in period t.** Recalling the timing of events described above, a worker that is successfully recruited in period t only begins working at the firm in period t+1 (and in the special case being analyzed here, **only** works for the firm in period t+1, because there is a probability equal to one (i.e., a 100 percent chance) that the individual will **not** be working for the firm in period t+2). In period t+1, the marginal output that the new recruit brings to the firm is $A_{t+1} f_n(k_{t+1}, n_{t+1})$ (i.e., the marginal product of labor in period t+1), and the wage the firm must pay the worker is w_{t+1}. Thus (using terminology that should be familiar from microeconomics), the **marginal revenue product** of the worker in period t+1 is $A_{t+1} f_n(k_{t+1}, n_{t+1}) - w_{t+1}$. But because the recruiting decisions of the firm are made in period t, this marginal revenue product is discounted by 1+*r*.

The left hand side of the condition reflects the marginal cost of recruiting. There is the direct cost J_t of recruiting; this cost is adjusted for the fact that recruiting only has a probability of success p_t^{FIND}.

Thus, the condition has the interpretation that the marginal cost of successful recruiting equals the (present value) marginal benefit of successful recruiting.

g. Consider the job creation condition in part e. As in part f, suppose that all workers turn over every period – that is, suppose

$$p_t^{TURNOVER} = p_{t+1}^{TURNOVER} = p_{t+2}^{TURNOVER} = p_{t+3}^{TURNOVER} = ...1.$$

In addition, suppose that a firm can **always** find a suitable worker – that is, suppose

$$p_t^{FIND} = p_{t+1}^{FIND} = p_{t+2}^{FIND} = p_{t+3}^{FIND} = ...1.$$

With these assumptions, what does the job creation condition simplify to? Briefly, but carefully, describe the economic interpretation of the job creation condition in this case?

Solution: Imposing $p_{t+1}^{TURNOVER} = 1$ **and** $p_t^{FIND} = p_{t+1}^{FIND} = 1$ on the job creation condition obtained in part e, we have

$$J_t = \left(\frac{1}{1+r}\right)\left(A_{t+1} f_n(k_{t+1}, n_{t+1}) - w_{t+1}\right).$$

The interpretation is identical to that in part f, except now when a firm begins the recruiting process by spending J_t, it knows that it will hire an employee for sure (because the probability of finding a worker is $p_t^{FIND} = 1$).

h. Analytically, can the job creation condition in part e be simplified so that it becomes **identical** to the labor demand condition studied in Chapter 6? **If so, describe the entire set of assumptions needed to make the two identical** (these assumptions would be of the form "variable *x* must have the numerical value *y*"). **If not, describe why there is no set of assumptions that makes the job creation condition identical to the labor demand condition. In either case, briefly and qualitatively describe the economics of why (or why not) the two conditions can be made to coincide. (Hint:** your analysis here may build on your analysis in part f and/or part g; if you do so, carefully explain how your analysis builds on part f and/or part g.)

Solution: The labor demand condition that emerged from the firm analysis in Chapter 6 was that the marginal product of labor equals the real wage **in every period**. In terms of the notation of this problem, that means $A_t f_n(k_t, n_t) = w_t$, $A_{t+1} f_n(k_{t+1}, n_{t+1}) = w_{t+1}$, $A_{t+2} f_n(k_{t+2}, n_{t+2}) = w_{t+2}$, ….

Inspecting the job creation condition in part e, as well as their successive simplifications in parts f and g, there is only simple assumption that, from a purely mathematical standpoint, makes the labor demand condition from Chapter 6 emerge: $J_t = J_{t+1} = J_{t+2} = ... = 0$. That is, suppose that recruiting costs were always zero. We are then left simply with $A_{t+1} f_n(k_{t+1}, n_{t+1}) = w_{t+1}$, which is the labor demand condition we studied earlier.

From a more conceptual economic standpoint, if all of the following three conditions are satisfied, the simple labor demand condition of Chapter 6 emerges: $p_t^{TURNOVER} = p_{t+1}^{TURNOVER} = p_{t+2}^{TURNOVER} = p_{t+3}^{TURNOVER} = ...1$ **and** $p_t^{FIND} = p_{t+1}^{FIND} = p_{t+2}^{FIND} = p_{t+3}^{FIND} = ...1$ **and** $J_t = J_{t+1} = J_{t+2} = ... = 0$. Imposing all of these restrictions on the job creation condition of part e, we are again left simply with $A_{t+1} f_n(k_{t+1}, n_{t+1}) = w_{t+1}$, which is the labor demand condition of Chapter 6.

From a purely mathematical point of view, though, all that is needed is $J_t = J_{t+1} = J_{t+2} = ... = 0$.

A much more general point emerges from this analysis: the standard (i.e., basic microeconomics) notion of demand (here, labor demand, but the more general point has nothing to do with labor markets per se) is a special case of the predictions of search theory. The standard notion of demand can be thought of as search activity in the special case that **if one searches, one will find what one is looking for with certainty** (in this case, searching for workers, but the point could be more general – say, searching for consumer goods); that **if one finds what one is looking for, one will only use/need it for one period** (rather than multiple time periods); and that **it is costless to search.**

Thus, search theory really is a generalization of the basic theory of supply and demand – generalization meaning that it allows one to consider all of the same issues (whether microeconomic or macroeconomic) for which supply/demand analysis is useful, but it also allows for considering richer issues (How long do workers stay at a job? How likely is it a suitable match will be found? Etc.). This generalization of the basic theory of supply and demand, which is the staple of all economic analysis, is part of the reason that Diamond, Mortensen, and Pissarides were awarded the Nobel.

Problem 4 continued

Consider again the job creation condition in part e, now ignoring all of the assumptions, analysis, and conclusions of part f, part g, and part h. S**uppose that the job creation condition reaches a steady state, in which all real variables *and all probabilities* (namely, p_t^{FIND} and $p_t^{TURNOVER}$) stop fluctuating from one time period to the next.** Construct **one single diagram** with (steady state) *w* plotted on the vertical axis and (steady state) *n* plotted on the horizontal axis to address the following:

i. Is the job creation condition upward-sloping, downward-sloping, perfectly horizontal, perfectly vertical, or is it impossible to tell? All that is needed is a qualitative sketch (numerical values are neither needed nor provided), and clearly label your diagram, providing any necessary explanation.

Solution: From a technical point of view, considering the steady state predictions of the job creation condition requires us to (as considering the steady state predictions of any of the frameworks we've studied) delete all of the time subscripts. Doing so gives us

$$\frac{J}{p^{FIND}} = \left(\frac{1}{1+r}\right)\left(Af_n(k,n) - w + \left(1 - p^{TURNOVER}\right)\frac{J}{p^{FIND}}\right)$$

This expression can be simplified by combining the terms J / p^{FIND} that appear on the left hand side and right hand side:

$$\frac{J}{p^{FIND}}\left[1 - \left(\frac{1 - p^{TURNOVER}}{1+r}\right)\right] = \left(\frac{1}{1+r}\right)\left(Af_n(k,n) - w\right).$$

The next couple of steps of algebra are simply rearrangements to enable ease of viewing this condition in the graphical space requested (*w* on the vertical axis, *n* on the horizontal axis). Combine terms in the square brackets on the left hand side:

$$\frac{J}{p^{FIND}}\left[\frac{r + p^{TURNOVER}}{1+r}\right] = \left(\frac{1}{1+r}\right)\left(Af_n(k,n) - w\right);$$

then multiply both sides by 1+*r*:

$$\left(r + p^{TURNOVER}\right)\frac{J}{p^{FIND}} = Af_n(k,n) - w;$$

finally, isolate the real wage *w*:

$$w = Af_n(k,n) - \left(r + p^{TURNOVER}\right)\frac{J}{p^{FIND}}..$$

We know that the marginal product of labor is decreasing in *n* (i.e., diminishing marginal product of labor). Though we do not have to, this is seen easily if we use the Cobb-Douglas functional form, for which $f_n(k,n) = (1-\alpha)k^{\alpha}n^{-\alpha} = (1-\alpha)\left(\frac{k}{n}\right)^{\alpha}$. Using this functional form, the previous expression becomes

$$w = A(1-\alpha)\left(\frac{k}{n}\right)^{\alpha} - \left(r + p^{TURNOVER}\right)\frac{J}{p^{FIND}}.$$

It is obvious from this way of expressing the (steady state version of the) job creation condition that there is an inverse relationship between *w* and *n*; thus, the job creation condition is downward sloping in the graphical space of *w* on the vertical axis and *n* on the horizontal axis (a qualitative sketch was all that was needed).

j. Starting from your sketch in part i, if the (steady state) probability of worker turnover rises (that is, if $p^{TURNOVER}$ increases), what happens to the job creation condition you plotted (i.e., does it rotate, shift, etc?)? All that is needed is a qualitative sketch (numerical values are neither needed nor provided), and clearly label your diagram, providing any necessary explanation.

Solution: The representation of the steady state version of the job creation condition in part i makes it easy to see what the shift factors are in the graphical space of *w* on the vertical axis and *n* on the horizontal axis. An increase in the rate of worker turnover shifts the job creation condition downwards. The economic intuition is that, for a given *n* (that is, for a given desired employment level), if workers are more likely to leave, a firm would be willing to pay a smaller wage due to the fact that the firm will have to incur recruiting costs more often, which eats into firm profits.

k. Starting from your sketch in part i, if the (steady state) probability of finding a suitable worker falls (that is, if p^{FIND} decreases), what happens to the job creation condition you plotted (i.e., does it rotate, shift, etc?)? All that is needed is a qualitative sketch (numerical values are neither needed nor provided), and clearly label your diagram, providing any necessary explanation.

Solution: The representation of the steady state version of the job creation condition in part i makes it easy to see what the shift factors are in the graphical space of *w* on the vertical axis and *n* on the horizontal axis. An increase in the likelihood of finding a suitable worker when a firm is recruiting makes it more willing to pay a higher wage, holding constant the level of desired employment. The economic intuition is similar to that above: with a higher probability of finding a suitable worker, a firm's job openings are unfilled for a **shorter** amount of time, which increases profits for the firm (i.e., jobs that go unfilled mean the firm is not able to produce and sell as much as it would like to).