Linearization, Log-Linearization, and Higher-Order Local (Level and Log) Approximations: Notes

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0.1 Non-Linear System

Write the non-linear conditions describing the solution of the original system as

$$f(x_t, y_t) = g(z_t). \tag{1}$$

As a concrete example, suppose $f(x_t, y_t) = x_t^{\sigma} + x_t y_t$ and $g(z_t) = z_t^{\nu}$.

0.2 Linearization in Levels

Let $\bar{x}$, $\bar{y}$, and $\bar{z}$ denote the steady-state values of x_t, y_t, and z_t, respectively (more generally, they are simply the values of the variables around which the local approximation is made). A first-order Taylor expansion around $(\bar{x}, \bar{y}, \bar{z})$ gives us

$$f(\bar{x}, \bar{y}) + f_x(\bar{x}, \bar{y})(x_t - \bar{x}) + f_y(\bar{x}, \bar{y})(y_t - \bar{y}) = g(\bar{z}) + g'(\bar{z})(z_t - \bar{z}). \tag{2}$$

Because $f(\bar{x}, \bar{y}) = g(\bar{z})$, these terms drop out. In our example, $f_x = \sigma x_t^{\sigma-1} + y_t$, $f_y = x_t$, and $g' = \nu z_t^{\nu-1}$, so

$$\left[\sigma \bar{x}^{\sigma-1} + \bar{y}\right](x_t - \bar{x}) + \bar{x}(y_t - \bar{y}) = \nu \bar{z}^{\nu-1}(z_t - \bar{z}) \tag{3}$$

is the linearized system in levels.

0.3 Linearization in Logs

Make a transformation of variables. Define $\hat{x}_t \equiv \ln x_t$, $\hat{y}_t \equiv \ln y_t$, $\hat{z}_t \equiv \ln z_t$. By construction, then, $x_t = \exp(\hat{x}_t)$. Rewrite the conditions describing the solution of the model as

$$f\left(\exp(\hat{x}_t), \exp(\hat{y}_t)\right) = g\left(\exp(\hat{z}_t)\right), \tag{4}$$

which is obviously identical to the solution of the original system. The functions f and g should now be thought of as functions of the transformed variables $\hat{x}_t$, $\hat{y}_t$, and $\hat{z}_t$. Denote by $\bar{\hat{x}}$ the steady-state value of $\hat{x}_t$, and similarly for $\bar{\hat{y}}$ and $\bar{\hat{z}}$. Clearly, $\bar{\hat{x}} = \ln \bar{x}$, etc. Now linearize this transformed system around the point $(\bar{\hat{x}}, \bar{\hat{y}}, \bar{\hat{z}})$, giving us

$$f\left(\bar{\hat{x}}, \bar{\hat{y}}\right) + f_{\hat{x}}\left(\bar{\hat{x}}, \bar{\hat{y}}\right)\left(\hat{x}_t - \bar{\hat{x}}\right) + f_{\hat{y}}\left(\bar{\hat{x}}, \bar{\hat{y}}\right)\left(\hat{y}_t - \bar{\hat{y}}\right) = g\left(\bar{\hat{z}}\right) + g'\left(\bar{\hat{z}}\right)\left(\hat{z}_t - \bar{\hat{z}}\right). \tag{5}$$

Because $f\left(\bar{\hat{x}}, \bar{\hat{y}}\right) = g\left(\bar{\hat{z}}\right)$, these terms drop out. The derivatives we need now are

$$f_{\hat{x}} = \sigma\left(\exp(\hat{x}_t)\right)^{\sigma} + \exp(\hat{x}_t)\exp(\hat{y}_t), \tag{6}$$

$$f_{\hat{y}} = \exp(\hat{x}_t)\exp(\hat{y}_t), \tag{7}$$

$$g' = \nu\left(\exp(\hat{z}_t)\right)^{\nu}. \tag{8}$$

Evaluating these derivatives at the point of expansion $(\bar{\hat{x}}, \bar{\hat{y}}, \bar{\hat{z}})$, the log-linearized system is

$$\left[\sigma\left(\exp(\bar{\hat{x}})\right)^{\sigma} + \exp(\bar{\hat{x}})\exp(\bar{\hat{y}})\right]\left(\hat{x}_t - \bar{\hat{x}}\right) + \exp(\bar{\hat{x}})\exp(\bar{\hat{y}})\left(\hat{y}_t - \bar{\hat{y}}\right) = \nu\left(\exp(\bar{\hat{x}})\right)^{\nu}\left(\hat{z}_t - \bar{\hat{z}}\right). \tag{9}$$

Expressing this equation in terms of the original variables x_t, y_t, z_t,

$$\left[\sigma\bar{x}^{\sigma} + \bar{x}\bar{y}\right]\left(\ln x_t - \ln\bar{x}\right) + \bar{x}\bar{y}\left(\ln y_t - \ln\bar{y}\right) = \nu\bar{z}^{\nu}\left(\ln z_t - \ln\bar{z}\right). \tag{10}$$

Equation (10) is the log-linearized approximation of (1), while equation (3) is the (level-)linearized approximation. In (10), it is often customary to define the terms $(\ln x_t - \ln \bar{x})$, $(\ln y_t - \ln \bar{y})$, and $(\ln z_t - \ln \bar{z})$ as $\hat{x}_t$, $\hat{y}_t$, $\hat{z}_t$. **Note that these hatted variables are not the same as the hatted variables we defined in the above derivations.**

0.4 Higher-Order Approximations

Second- and higher-order approximations in both levels and logs are now straightforward to obtain. For example, to obtain the quadratic approximation in levels, simply take a second-order Taylor expansion of (1) around $(\bar{x}, \bar{y}, \bar{z})$. To obtain a log-quadratic approximation, take a second-order Taylor expansion of (4) around $(\bar{\hat{x}}, \bar{\hat{y}}, \bar{\hat{z}})$. And so on for higher-order approximations.