Department of Economics University of Maryland

Economics 701
Advanced Macroeconomics I
Project 1 Suggested Solutions
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For the transformed (i.e., stationary) model, the social planning problem is

$$\max_{\{c_t,n_t,k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} b^t X_t^{1-\sigma} \left\{ u(c_t,n_t) + \lambda_t \left[z_t f(k_t,n_t) - c_t - \gamma_t k_{t+1} + (1-\delta)k_t \right] \right\}$$

taking as given the exogenous law of motions governing (high-frequency) TFP fluctuations z_t ,

$$\ln z_{t+1} = (1-\rho_z)\ln \bar{z} + \rho_z \ln z_t + \varepsilon_{t+1}^z ,$$

and (low-frequency) trend growth fluctuations,

$$\ln \gamma_{t+1} = (1-\rho_\gamma)\ln \bar{\gamma} + \rho_\gamma \ln \gamma_t + \varepsilon_{t+1}^\gamma .$$

The first-order conditions with respect to c_t, n_t, and k_{t+1} are, respectively, $u_{ct} - \lambda_t = 0$, $u_{nt} + \lambda_t z_t f_{nt} = 0$, and

$$-b^t X_t^{1-\sigma}\lambda_t\gamma_t + E_t\left\{ b^{t+1} X_{t+1}^{1-\sigma}\lambda_{t+1}\left[z_{t+1} f_{kt+1} + 1 - \delta \right] \right\} = 0 .$$

Combining first-order conditions and rearranging, we have the consumption-leisure efficiency condition

$$-\frac{u_n(c_t,n_t)}{u_c(c_t,n_t)} = z_t f_{nt} .$$

For the consumption-investment efficiency condition (Euler equation), recognize that, by assumption, $\gamma_t \equiv \frac{X_{t+1}}{X_t}$ is in the information set of period *t*. Hence, the Euler equation is

$$\gamma_t = \gamma_t^{1-\sigma} E_t\left\{ \frac{b u_{ct+1}}{u_{ct}} \left[z_t f_{kt+1} + 1 - \delta \right] \right\} .$$

Efficient allocations are thus a set of endogenous state-contingent processes $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$ characterized by three state-contingent efficiency conditions: the sequence of consumption-leisure conditions, the sequence of Euler equations (which is replaced by a transversality condition as $T \to \infty$), and the sequence of resource constraints

$$c_t + \gamma_t k_{t+1} - (1-\delta)k_t = z_t f(k_t, n_t).$$

Note that in the Euler equation, if $\gamma = 1 \ \forall t$, then the "usual" Euler equation (i.e., from the "standard" RBC model with growth completely ignored, even though growth **is** in the background in the form of the functional form restrictions it imposes) emerges.

Using the given functional forms for utility

$$u(c_t, n_t) = \frac{\left[c_t(1-n_t)^{\psi}\right]^{1-\sigma} - 1}{1-\sigma}$$

and production

$$f(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha},$$

(both of which were used in constructing the stationary version of the problem), the value of ψ needed in each of three parameter sets in order to make $\bar{n} = 0.3$ in each of the three steady states is given in the table:

	Parameter Set A (Baseline)	**Parameter Set B (Constant Growth)**	**Parameter Set C (Variable Growth)**
$\bar{\gamma}$	1	$1.03^{1/4}$	$1.03^{1/4}$
b	0.99	0.99	0.99
δ	0.02	0.02	0.02
σ	1.5	1.5	1.5
α	0.36	0.36	0.36
ψ	**1.962**	**1.962**	**1.962**
ρ_z	0.92	0.92	0.92
σ_z	0.006	0.006	0.006
ρ_γ	---	---	0.99
σ_γ	---	---	0.001; 0.003; 0.01

Frisch Elasticity of Labor Supply

To calculate the Frisch elasticity of labor supply, use the consumption-leisure optimality condition **from the perspective of individual households** – that is, $-\frac{u_n(c_t,n_t)}{u_c(c_t,n_t)} = w_t$. For the given functional for utility, this can be expressed as

$$\psi c_t = w_t(1-n_t).$$

Computing the elasticity of labor supply with respect to the real wage thus requires computing $\frac{\partial n_t}{\partial w_t}\frac{w_t}{n_t}$. Use the implicit function theorem (or brute force algebra) to compute the required partial; the Frisch elasticity is thus $\frac{1-n_t}{n_t}$, which in the steady state (given you had to calibrate so that $n^{SS} = 0.3$) is 0.7/0.3 ≈ 2.33. For the given utility function, clearly the elasticity fluctuates over time (as labor fluctuates), so one could also have calculated the business-cycle fluctuations of the Frisch elasticity.

Numerical Approximation

Using the notation of SGU (2004), one representation of the model is to organize the endogenous and exogenous variables into the state vector $x_t = \begin{bmatrix} k_t \\ \ln z_t \\ \ln \gamma_t \end{bmatrix}$ and co-state vector $y_t = \begin{bmatrix} c_t \\ n_t \end{bmatrix}$. You could have eliminated either or both consumption or labor from the model, in which case you would have a different (or no) co-state vector at all. But the state vector cannot be anything different.

Let's first consider the case of zero persistence in shocks to long-run growth in the context of parameter set A (this persistence is of course immaterial in parameter set A because you are told that there are no shocks to growth, but this allows for easy nesting of the parameter set A without having to redefine the state vector when moving across parameter sets).

Conducting a first-order approximation in levels (you may have used a log-linear approximation, as is also common) to the equilibrium of the model using the SGU algorithm, the first-order accurate decision rules for parameter set A are

$$g_x = \begin{bmatrix} 0.0330 & 0.3601 & 0.0738 \\ -0.0042 & 0.2319 & -0.0304 \end{bmatrix}, \quad h_x = \begin{bmatrix} 0.9663 & 1.4514 & -14.6491 \\ 0 & 0.92 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For parameter sets B and C, the same approximation procedure, conditional on zero persistence in growth fluctuations, leads to

$$g_x = \begin{bmatrix} 0.0425 & 0.4061 & -0.1168 \\ -0.0054 & 0.1801 & 0.0576 \end{bmatrix}, \; h_x = \begin{bmatrix} 0.9600 & 0.9890 & -8.5781 \\ 0 & 0.92 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note this latter pair of g_x and h_x matrices is identical across parameter constellations B and C (again, conditional on $\rho_\gamma = 0$). Furthermore, note that all of the first-order accurate matrices are independent of the standard deviation of the innovations to the exogenous processes due to the certainty equivalence inherent in conducting a first-order approximation.

Results

Parameter Set A corresponds to the "standard" zero-growth RBC model; as such, we should see similar relative volatilities, autocorrelations, and cross-correlations as in, say, King and Rebelo (1999). Broadly, we do (keep in mind that we are using different, though illustratively similar, parameters than in King and Rebelo (1999), for things such as the separability of preferences, the persistence and standard deviation of TFP, etc.)

For the constant positive growth case, the dynamics are very little changed, even though the steady state quantities of consumption, investment, and output are different. However, recall that the **levels** of quantity variables denominated in consumption units do not have any interpretation. It is only ratios of such variables that matter.

Parameter Set A (Baseline) – HP filtered simulations

	gdp	**c**	**inv**	**labor**	**Z**
Mean	1.212	0.922	0.290	0.300	1
Long-run share (/ mean(gdp))	1	0.761	0.239	---	---
Volatility (SD%)	1.120	0.317	3.766	0.583	0.749
Relative volatility (/gdp)	1	0.284	3.363	0.521	0.669
Autocorrelation	0.678	0.746	0.672	0.671	0.674
Correlation with gdp	1	0.930	0.994	0.990	0.998

Parameter Set B (Constant Growth) – HP filtered simulations

	gdp	**c**	**inv**	**labor**	**z**
Mean	1.013	0.771	0.242	0.300	1
Long-run share (/ mean(gdp))	1	0.761	0.239	---	---
Volatility (SD%)	1.039	0.416	3.094	0.453	0.749
Relative volatility	1	0.400	2.977	0.436	0.721

(/gdp)					
Autocorrelation	0.681	0.728	0.672	0.669	0.674
Correlation with gdp	1	0.964	0.993	0.985	0.997

For the variable growth cases, however, dynamics depend critically on the persistence and standard deviation of shocks to long run growth (perhaps not surprisingly), as the follow sequence of tables shows.

Parameter Set C (Variable Growth, $\rho_\gamma = 0,\ \sigma_\gamma = 0.001$) – HP filtered simulations

	gdp	**c**	**inv**	**labor**	**z**
Mean	1.013	0.771	0.242	0.300	1
Long-run share (/ mean(gdp))	1	0.761	0.239	---	---
Volatility (SD%)	1.041	0.422	3.095	0.453	0.749
Relative volatility (/gdp)	1	0.405	2.975	0.435	0.720
Autocorrelation	0.681	0.730	0.670	0.669	0.674
Correlation with gdp	1	0.959	0.991	0.983	0.997

Parameter Set C (Variable Growth, $\rho_\gamma = 0.50,\ \sigma_\gamma = 0.001$) – HP filtered simulations

	gdp	**c**	**inv**	**labor**	**z**
Mean	1.013	0.771	0.242	0.300	1
Long-run share (/ mean(gdp))	1	0.761	0.239	---	---
Volatility (SD%)	1.041	0.430	3.095	0.453	0.749
Relative volatility (/gdp)	1	0.413	2.971	0.435	0.720
Autocorrelation	0.681	0.737	0.671	0.670	0.674
Correlation with gdp	1	0.948	0.989	0.977	0.996

Parameter Set C (Variable Growth, $\rho_\gamma = 0.95,\ \sigma_\gamma = 0.001$) – HP filtered simulations

	gdp	**c**	**inv**	**labor**	**z**
Mean	1.013	0.771	0.242	0.300	1
Long-run share (/ mean(gdp))	1	0.761	0.239	---	---
Volatility (SD%)	1.050	0.489	3.197	0.476	0.749
Relative volatility (/gdp)	1	0.466	3.043	0.453	0.714
Autocorrelation	0.686	0.770	0.677	0.680	0.674
Correlation with gdp	1	0.857	0.962	0.928	0.987

Parameter Set C (Variable Growth, $\rho_\gamma = 0,\ \sigma_\gamma = 0.005$) – HP filtered simulations

	gdp	c	inv	labor	z
Mean	1.013	0.771	0.242	0.300	1
Long-run share (/ mean(gdp))	1	0.761	0.239	---	---
Volatility (SD%)	1.112	0.714	3.502	0.500	0.749
Relative volatility (/gdp)	1	0.642	3.150	0.450	0.674
Autocorrelation	0.655	0.786	0.537	0.652	0.674
Correlation with gdp	1	0.776	0.822	0.774	0.936

Parameter Set C (Variable Growth, $\rho_\gamma = 0.50,\ \sigma_\gamma = 0.005$) – HP filtered simulations

	gdp	c	inv	labor	z
Mean	1.013	0.771	0.242	0.300	1
Long-run share (/ mean(gdp))	1	0.761	0.239	---	---
Volatility (SD%)	1.184	1.080	4.186	0.552	0.749
Relative volatility (/gdp)	1	0.912	3.537	0.466	0.633
Autocorrelation	0.716	0.860	0.582	0.744	0.674
Correlation with gdp	1	0.758	0.560	0.451	0.880

Parameter Set C (Variable Growth, $\rho_\gamma = 0.95,\ \sigma_\gamma = 0.005$) – HP filtered simulations

	gdp	c	inv	labor	z
Mean	1.013	0.771	0.242	0.300	1
Long-run share (/ mean(gdp))	1	0.761	0.239	---	---
Volatility (SD%)	1.535	1.386	55.204	0.827	0.749
Relative volatility (/gdp)	1	0.903	35.961	0.539	0.488
Autocorrelation	0.782	0.875	0.685	0.727	0.674
Correlation with gdp	1	0.610	0.331	0.353	0.760

Band-Pass Filtered Simulations

Running the same simulations through the band-pass filter, rather than the HP filter, the following are the analogs of the previous set of tables. In these results, the parameters [6,32] are used in running the BP filter.

Parameter Set A (Baseline) – BP filtered simulations

	gdp	**c**	**inv**	**labor**	**Z**
Mean	1.212	0.922	0.290	0.300	1
Long-run share (/ mean(gdp))	1	0.761	0.239	---	---
Volatility (SD%)	0.944	0.310	3.009	0.452	0.658
Relative volatility (/gdp)	1	0.328	3.189	0.479	0.697
Autocorrelation	0.886	0.893	0.885	0.885	0.886
Correlation with gdp	1	0.973	0.996	0.994	0.998

Parameter Set B (Constant Growth) – BP filtered simulations

	gdp	**c**	**inv**	**labor**	**z**
Mean	1.013	0.771	0.242	0.300	1
Long-run share (/ mean(gdp))	1	0.761	0.239	---	---
Volatility (SD%)	0.911	0.360	2.728	0.398	0.658
Relative volatility (/gdp)	1	0.395	2.995	0.437	0.722
Autocorrelation	0.887	0.895	0.885	0.885	0.886
Correlation with gdp	1	0.969	0.993	0.988	0.998

Parameter Set C (Variable Growth, $\rho_\gamma = 0$, $\sigma_\gamma = 0.001$) – BP filtered simulations

	gdp	c	inv	labor	z
Mean	1.013	0.771	0.242	0.300	1
Long-run share (/ mean(gdp))	1	0.761	0.239	---	---
Volatility (SD%)	0.912	0.366	2.727	0.398	0.658
Relative volatility (/gdp)	1	0.401	2.989	0.436	0.721
Autocorrelation	0.887	0.896	0.884	0.884	0.886
Correlation with gdp	1	0.962	0.992	0.985	0.997

Parameter Set C (Variable Growth, $\rho_\gamma = 0.50$, $\sigma_\gamma = 0.001$) – BP filtered simulations

	gdp	c	inv	labor	z
Mean	1.013	0.771	0.242	0.300	1
Long-run share (/ mean(gdp))	1	0.761	0.239	---	---
Volatility (SD%)	0.913	0.375	2.727	0.399	0.658
Relative volatility (/gdp)	1	0.410	2.986	0.436	0.720
Autocorrelation	0.887	0.897	0.884	0.885	0.886
Correlation with gdp	1	0.949	0.989	0.978	0.996

Parameter Set C (Variable Growth, $\rho_\gamma = 0.95$, $\sigma_\gamma = 0.001$) – BP filtered simulations

	gdp	c	inv	labor	z
Mean	1.013	0.771	0.242	0.300	1
Long-run share (/ mean(gdp))	1	0.761	0.239	---	---
Volatility (SD%)	0.920	0.417	2.827	0.419	0.658
Relative volatility (/gdp)	1	0.453	3.073	0.455	0.715
Autocorrelation	0.888	0.902	0.886	0.888	0.886
Correlation with gdp	1	0.860	0.963	0.935	0.988

References

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