The Basic RBC Model: Theory and Computation

1 The Model

Here we sketch the basic RBC model with four exogenous processes: TFP z_t; government spending g_t; a proportional labor income tax rate τ_t^n; and a proportional capital income tax rate τ_t^k. The infinitely-lived representative household's problem is to maximize lifetime utility

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \tag{1}$$

subject to the sequence of flow budget constraints

$$c_t + k_{t+1} + T_t = (1 - \tau_t^n) w_t n_t + \left[1 + (1 - \tau_t^k)(r_t - \delta)\right] k_t, \tag{2}$$

where w_t is the real wage rate, r_t is the real capital rental rate, δ is the depreciation rate of capital, and τ_t^n and τ_t^k are tax rates on labor and capital income, respectively. The household chooses in each period consumption c_t, labor n_t, and next-period capital holdings k_{t+1}, taking as given all other variables. Taxes finance exogenous government spending; the government budget constraint is given by

$$\tau_t^n w_t n_t + \tau_t^k (r_t - \delta) k_t + T_t = g_t. \tag{3}$$

We allow for a lump-sum component of taxes, T_t, in order to abstract from government financing issues, as the basic RBC model does.

Summing the flow household budget constraint and the flow government budget constraint yields, as always, the economy's resource frontier (aka, the PPF),

$$c_t + k_{t+1} - (1 - \delta) k_t + g_t = z_t f(k_t, n_t), \tag{4}$$

where $f(k_t, n_t)$ is the production technology of the economy that is buffeted by shocks to the Solow process (aka TFP) z_t. The production technology is operated by a continuum of identical perfectly-competitive firms (equivalently, a single representative firm), who hire labor and capital in perfectly-competitive spot markets, taking as given w_t and r_t. Flow profits of the firm are $z_t f(k_t, n_t) - w_t n_t - r_t k_t$, and profit-maximization by firms leads to the standard conditions that factor prices (i.e., the wage and the rental rate) equal their respective marginal products: $w_t = z_t f_n(k_t, n_t)$

and $r_t = z_t f_k(k_t, n_t)$. For a complete exposition of the basic RBC model, see the seminal work of Kydland and Prescott (1982), the overview by King and Rebelo (1999), or any of the by-now countless descriptions of the RBC model.

Each of the four exogenous processes evolves according to an AR(1) in logs:

$$\ln z_{t+1} = (1-\rho_z)\bar{z} + \rho_z \ln z_t + \epsilon^z_{t+1}, \tag{5}$$

$$\ln g_{t+1} = (1-\rho_g)\bar{g} + \rho_g \ln g_t + \epsilon^g_{t+1}, \tag{6}$$

$$\ln \tau^n_{t+1} = (1-\rho_{\tau^n})\bar{\tau^n} + \rho_{\tau^n} \ln \tau^n_t + \epsilon^{\tau^n}_{t+1}, \tag{7}$$

$$\ln \tau^k_{t+1} = (1-\rho_{\tau^k})\bar{\tau^k} + \rho_{\tau^k} \ln \tau^k_t + \epsilon^{\tau^k}_{t+1}. \tag{8}$$

Each of the ρ_{xx} parameters is the AR(1) coefficient of the respective process; each of the $\bar{xx}$ variables is the log-mean of the exogenous variable; each of the ϵ^{xx}_{t+1} is the shock to process xx; and for simplicity we assume that the processes are uncorrelated with each other. This assumption, and, indeed, the assumption of AR(1) processes for each of these driving forces, are made for illustrative purposes; the goal here is to see the basic model in operation computationally. The time-t realizations of the shocks ϵ^{xx}_t are known by all agents in the economy at the very beginning of time t, before time-t decisions are made.

2 Equilibrium

Because we are allowing for distortionary taxes, the first welfare theorem fails, and we cannot appeal to a Social Planner problem in order to characterize the equilibrium of the model. We must therefore resort to solving for the equilibrium directly using the household and firm first-order conditions along with the resource frontier. Standard Lagrangian (or value-function) techniques yield the household optimality conditions,

$$-\frac{u_n(c_t, n_t)}{u_c(c_t, n_t)} = (1-\tau^n_t)w_t \tag{9}$$

$$u_c(c_t, n_t) = E_t\left\{\beta u_c(c_{t+1}, n_{t+1})\left[1 + (1-\tau^k_{t+1})(r_{t+1}-\delta)\right]\right\}, \tag{10}$$

in which the time-t conditional expectations operator $E_t(.)$ appears because we allow for sequentially-unfolding uncertainty in the four exogenous processes (as described above). Condition (9) describes the household's optimal consumption-leisure tradeoff: at the optimum, the marginal rate of substitution (MRS) between consumption and leisure is equated to the after-tax wage. Condition (10) describes the household's optimal consumption-savings tradeoff: at the optimum, the MRS between

period-t consumption and period-$t+1$ consumption is equated to the after-tax return on savings (capital holdings).[1]

Along with the firm optimality conditions, the four laws of motion for the four exogenous processes, and the resource constraint, conditions (9) and (10) describe the equilibrium of the model. If we substitute the firm optimality conditions ($w_t = z_t f_n(k_t, n_t)$ and $r_t = z_t f_k(k_t, n_t)$) into (9) and (10), the equilibrium of the model is described by the three conditions (4), (9), (10), and the three endogenous variables to be solved for are c_t, n_t, and k_{t+1}.

3 Computing the Equilibrium

In any given period t, the choice variables of the model are c_t, n_t, and k_{t+1}. In general, we cannot solve this model analytically; we must resort to computational methods. Saying we must "resort to computational methods" to solve the model already raises a host of questions. In particular, *what does this even mean?* In practice (and this is even before we get to computations), this means specifying a solution for each endogenous variable as a *function* of the *state variables* of the model. Remember that the RBC model is a dynamic model — variables evolve over time. Loosely speaking, the *state* of the model at a given point in time is the current and past realizations of what has happened in the model. Briefly, we could say that the state of the model is the history of what has already transpired. In the context of solving the model, a state variable is one that is associated with the model, rather than being associated with the optimization problem of the household or firm.

Let's consider the *state* of the model in more detail. In principle, the state of the model at any point in time t consists of the period-t realizations of the exogenous variables, the time-t quantities of any endogenous accumulation (stock) variables (in the basic RBC model, capital, because the period-t capital stock is "inherited" from period $t-1$), along with *their entire histories.* That is (assuming perfect record-keeping and perfect recall, etc), in period t, the, say, period $t-14$ realization of the labor tax rate τ^n_{t-14} *is* known, so in principle the period-t consumption, labor, or capital decisions could be functions of it. A convenient technical consequence, though, of specifying the exogenous processes as AR(1)s is that period-t decisions are functions of *only* the period-t state variables.[2] That is, c_t, n_t, and k_{t+1} are functions only of $[k_t, z_t, g_t, \tau^n_t, \tau^k_t]$. From our, the modeler's,

[1] Recall that any fully-fledged general equilibrium macro model has three types of markets: goods markets, labor markets, and asset (savings) markets. The representative household's optimal tradeoffs between its supplies and demands in these markets are described by (9) and (10). Again, see Kydland and Prescott (1982), King and Rebelo (1999), or any description of the basic RBC model for more on the derivations of these conditions.

[2] This is akin to the notion of a *Markov matrix* in linear algebra. In the parlance of linear algebra, a Markov process is one in which the evolution of a vector of variables is described by repeated multiplication by the same matrix (known as the transition matrix). In order to determine the next value of the vector of variables in the chain

perspective, then, the state variables of the model are $[k_t, z_t, g_t, \tau_t^n, \tau_t^k]$. Let's summarize this vector as $\mathbf{S_t}$. What "solving the model" now means is that we are trying to find functions $c(\mathbf{S_t})$, $n(\mathbf{S_t})$, $k(\mathbf{S_t})$ that describe how the endogenous variables depend on $\mathbf{S_t}$.

These functions cannot be determined analytically; so now the "solving computationally" part comes in. More precisely, what we must do is numerically *approximate* these *unknown* (and ultimately, unknowable...) functions. In the simplest possible terms, what we end up doing is constructing, in ways to be described next, functions $c^{approx}(\mathbf{S_t})$, $n^{approx}(\mathbf{S_t})$, $k^{approx}(\mathbf{S_t})$, that we *believe* (hope?) are "close enough" to the true, unknowable actual functions $c(\mathbf{S_t})$, $n(\mathbf{S_t})$, $k(\mathbf{S_t})$.

There are a great many algorithms, borrowed from the rich field of numerical analysis in mathematics, that one can employ to approximate these unknown functions. A useful distinction to make between the many algorithms is that between *global approximations* and *local approximations.*

3.1 Global Approximations

As the name implies, a global approximation method is one in which the technique used leads to approximations $c^{approx}(\mathbf{S_t})$, $n^{approx}(\mathbf{S_t})$, $k^{approx}(\mathbf{S_t})$ that are (hopefully...) very close to the true $c(\mathbf{S_t})$, $n(\mathbf{S_t})$, $k(\mathbf{S_t})$ over a very broad ("global") range of possible values of the state vector $\mathbf{S_t}$. The obvious pro of such an approach is that the approximation should be fairly accurate no matter where the model "is" in the unfolding of its dynamic events. A con of this approach is that, given current hardware and software limitations, these methods can only be practically applied to relatively small models.[3] Of course, it's likely that in N years, computational power will increase sufficiently that global methods will eventually become the default way to do things.[4] For an overview of some of the most popular (popular because they have proven sufficiently useful) global methods in use today, see Aruoba, Fernandez-Villaverde, and Rubio-Ramirez (2006).

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of vectors (termed, surprisingly enough, the Markov chain), all that need be done is multiply the transition matrix by the current vector of variables in the chain. Knowledge of past vectors in the chain is not needed. AR(1) processes have representations as Markov processes. Thus (this argument is not completely technically correct, but the idea is right...), in a model driven by AR(1) shocks, a so-called "Markov-equilibrium" of the model can usually be found. A technical point that we will not consider: for a lot of models, Markov equilibria are only one type of equilibrium that arises. Non-Markov equilibria, in which the relevant period-t state *does* include lagged variables, also can arise. As a general rule, though, easy-to-apply algorithms (conceptually as well as technically) do not exist for finding non-Markov equilibria.

[3] For example, the model of Christiano, Eichenbaum, and Evans (2005) would probably very hard (impossible?) to solve using global methods given current technology. Likewise, the Board's domestic and international models would also likely be impossible to solve using global methods.

[4] To be clear, by today's standards the basic RBC model is considered a tiny model. As such, it easily yields to global approximations.

3.2 Local Approximations

Because we probably want to be able to solve models today, though, and not have to wait N years, local approximations are likely the only way to go medium- and large-scale macro models. Local approximations are all based on Taylor's theorem and Taylor approximations, which you are familiar with from basic calculus. Recall that Taylor approximations are only *locally* accurate: a Taylor approximation is a linear or higher-order approximation to some function *around a given point*, and the accuracy of the approximation deteriorates as one moves further away from that point. This feature is in contrast to global approximations, which by construction do not deteriorate in accuraxy as the model moves away from any given point.

There are a great many specific algorithms that implement first-order (linear) approximations, which are of course the simplest type of Taylor approximation. In the end, they all boil down to solving a linear system of equations, for which there are in general a great many techniques in linear algebra. We will use the implementation by Schmitt-Grohe and Uribe (2004). As Schmitt-Grohe and Uribe discuss, once one has a linear approximation in hand, it can be relatively straightforward to implement higher-order (second-order, third-order, etc.) approximations. There are not as many general algorithms for generating second-order approximations as there are for generating first-order approximations, partly because until only relatively recently, in the realm of local approximations, first-order solutions were thought to be "good enough" for most applications, and partly because computational power has advanced enough to make second-order approximations of moderately-big models only recently practical.[5]

3.2.1 Steady-State

Having decided we will focus on a local approximation, we must decide around which point of the model we wish to approximate. A natural candidate is the *steady-state* of the model. The notion of a steady state is one from the theory of differential equations. Recall that a differential equation (or, its discrete-time counterpart, a difference equation) is one in which variables at different points in time are related to each other somehow. Notice that equation (10) is just such an equation: it relates consumption and leisure in period t to consumption and leisure in period $t+1$. It is thus a difference equation. When a difference equation is allowed to evolve for a very long period of time (technically, for an infinite number of periods), it may (or may not...) settle down to a fixed point. Alternatively, a difference process may begin at a certain point and may, for some reason, never be perturbed from that point. In either case, the point (either reached eventually or from which

[5]For certain types of welfare comparisons (as in policy experiments), it turns out first-order approximations can give misleading results, whereas second-order and higher approximations do not. This issue is beyond our scope for now, though, and does not arise in solving and simulating the basic RBC model.

it never moves at all) is termed a *steady state* of the difference process. Very loosely speaking, the steady-state of a difference process is one in which time does not matter at all or has no meaning.

In our model, condition (10) is a difference process. To speak properly of a steady-state in our model, we first shut down the exogenous processes, so that the steady-state we are considering is the non-stochastic steady-state. In the non-stochastic steady-state (when time "no longer matters," so to speak), we can write (10) as

$$u_c(c,n) = \beta u_c(c,n)\left[1 + (1-\tau^k)(f_k(k,n) - \delta)\right]. \tag{11}$$

Note that, mechanically, all we have done is remove the expectations operator (because we have shut down the shocks of the model) and removed all time subscripts (because time "doesn't matter" in the steady-state). Imposing steady-state on (4) and (9), we end up with three equations in the three *numbers* (not sequences) c, n, k, which we can solve using any routine numerical-zero-finder. The resulting vector (c^{ss}, n^{ss}, k^{ss}) is the steady-state of the model, and is the steady-state around which we conduct our local approximation.

References

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